Divide-and-Conquer

- **Divide** the problem into a number of sub-problems
  – Similar sub-problems of smaller size
- **Conquer** the sub-problems
  – Solve the sub-problems recursively
  – Sub-problem size small enough $\Rightarrow$ solve the problems in straightforward manner
- **Combine** the solutions of the sub-problems
  – Obtain the solution for the original problem

Merge Sort Approach

- To sort an array $A[p \,..\, r]$:
- **Divide**
  - Divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each
- **Conquer**
  - Sort the subsequences recursively using merge sort
  - When the size of the sequences is 1 there is nothing more to do
- **Combine**
  - Merge the two sorted subsequences

Merge Sort

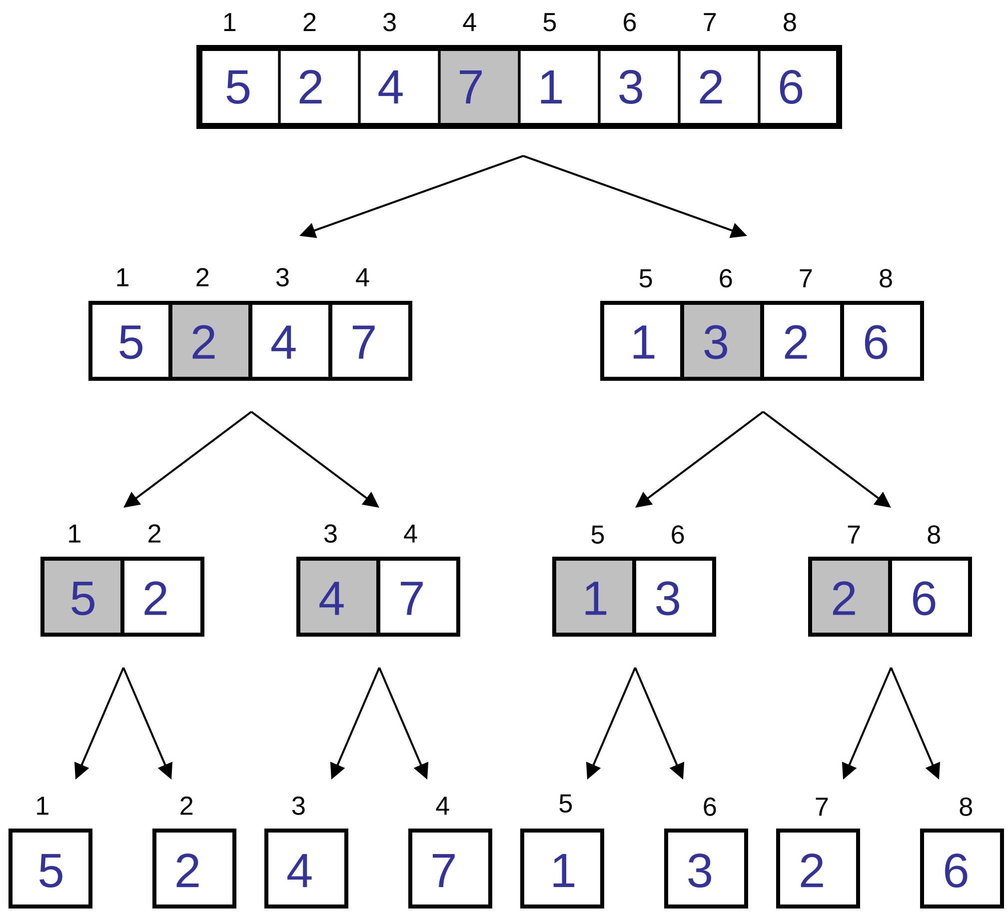
```
Algorithm MergeSort(low, high)
// a[low : high] is a global array to be sorted.
// Small(P) is true if there is only one element
// to sort. In this case the list is already sorted.
{
    if (low < high) then  // If there are more than one
    {
        // Divide P into subproblems.
            // Find where to split the set.
                mid := ⌊(low + high)/2⌋;
        // Solve the subproblems.
            MergeSort(low, mid);
            MergeSort(mid + 1, high);
        // Combine the solutions.
            Merge(low, mid, high);
    }
}
```

Merging two sorted sub-arrays

```
Algorithm Merge(low, mid, high)
// a[low : high] is a global array containing two sorted
// subsets in a[low : mid] and in a[mid + 1 : high]. The goal
// is to merge these two sets into a single set residing
// in a[low : high]. b[ ] is an auxiliary global array.
{
    h := low; i := low; j := mid + 1;
    while ((h ≤ mid) and (j ≤ high)) do
    {
        if (a[h] ≤ a[j]) then
        {
            b[i] := a[h];h := h + 1;
        }
        else
        {
            b[i] := a[j]; j := j + 1;
        }
        i := i + 1;
    }
    if (h > mid) then
        for k := j to high do
        {
            b[i] := a[k]; i := i + 1;
        }
    else
        for k := h to mid do
        {
            b[i] := a[k]; i := i + 1;
        }
    for k := low to high do a[k] := b[k];
}
```

Example – *n* Power of 2

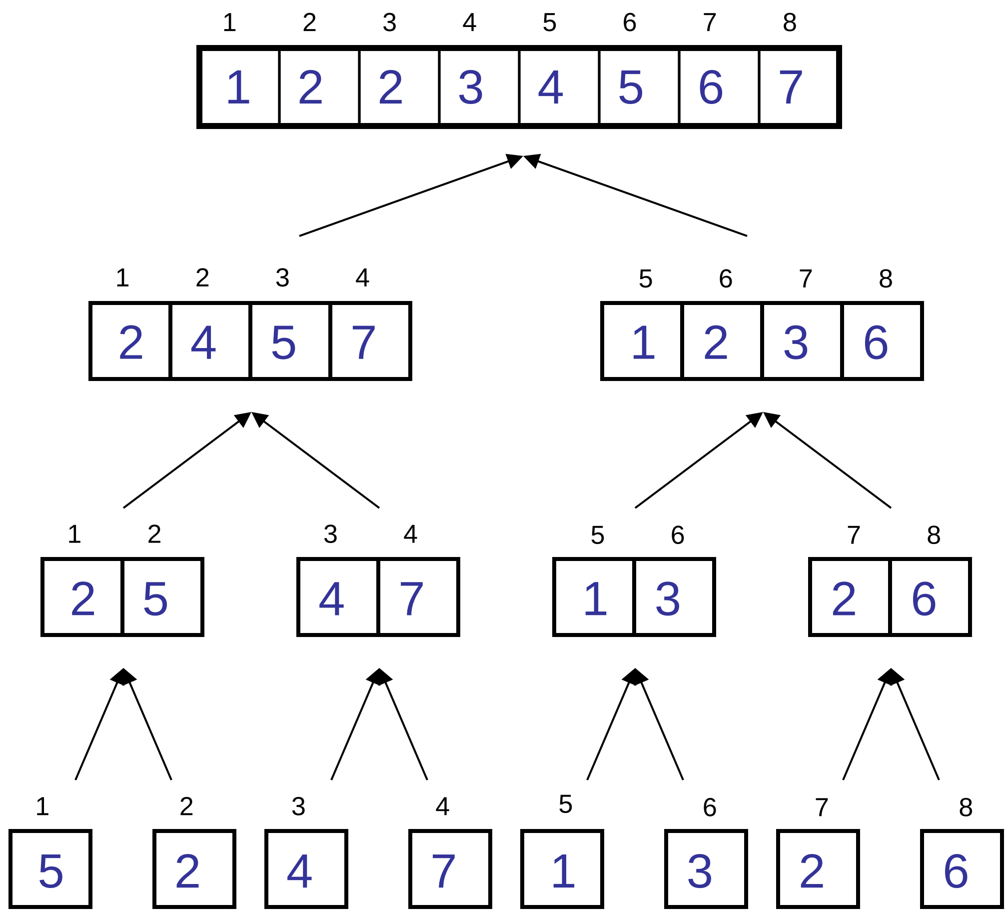
Divide



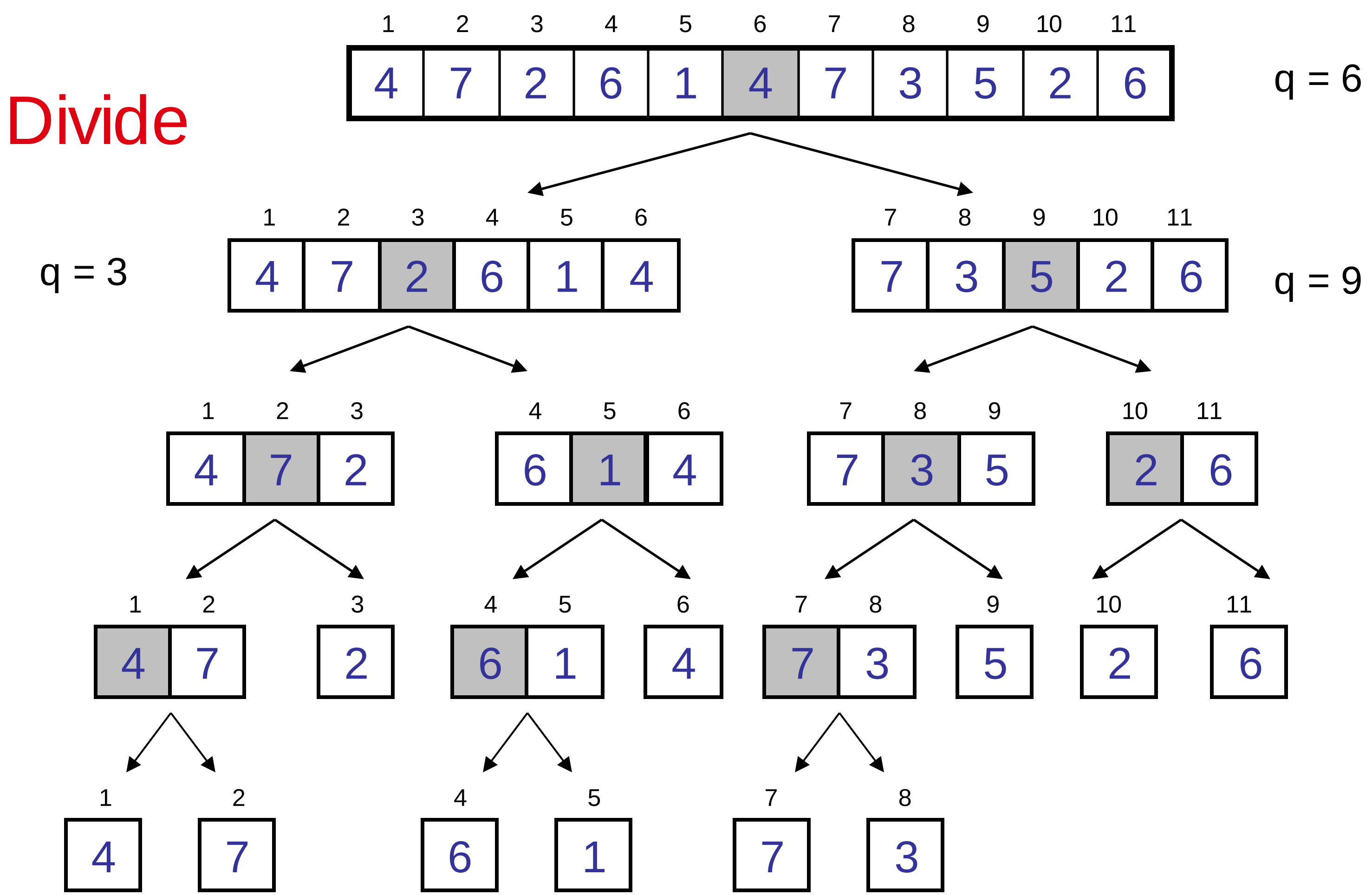
q = 4

Example – *n* Power of 2

Conquer
and
Merge

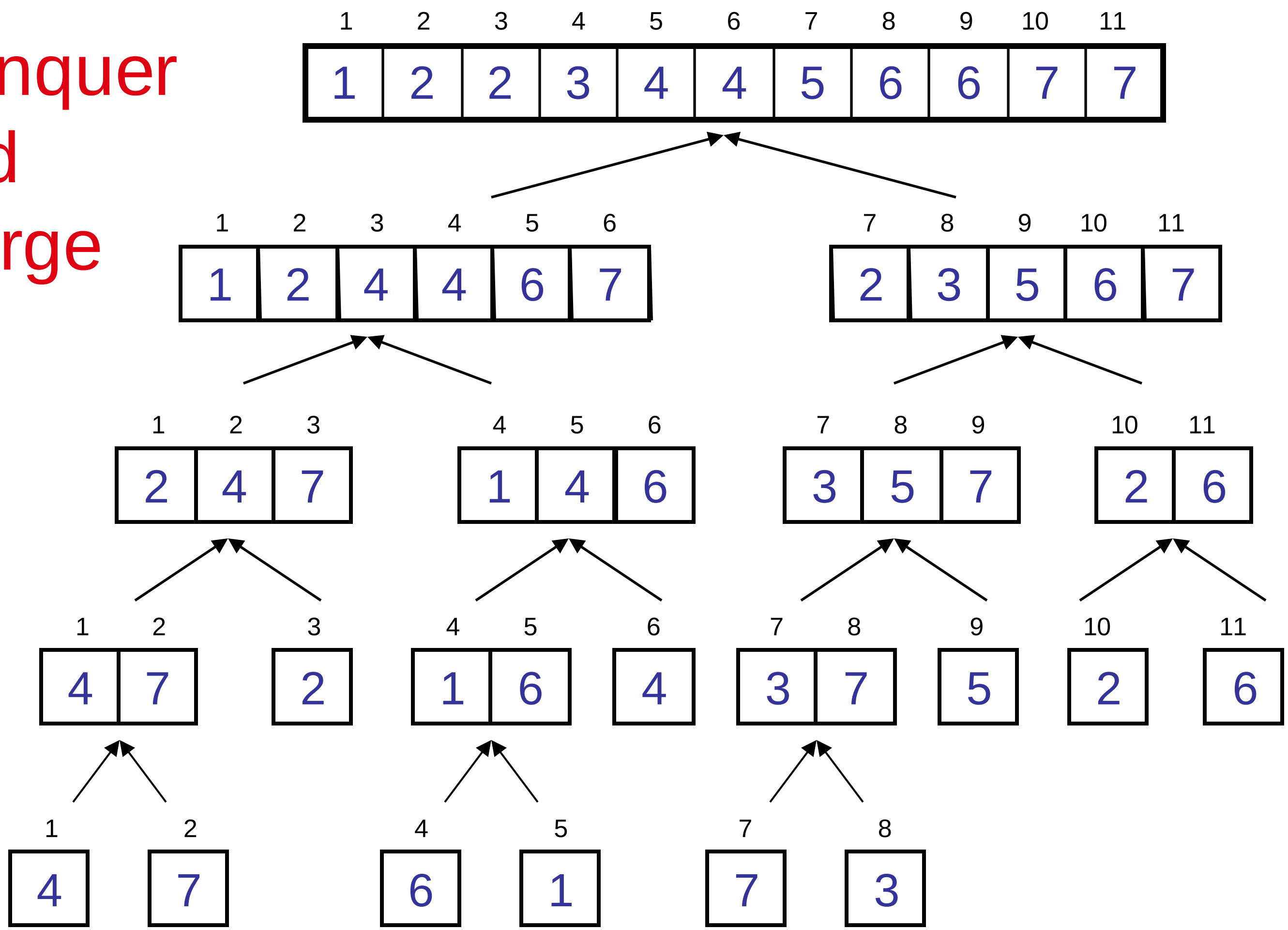


Example – n Not a Power of 2



Example – n Not a Power of 2

Conquer
and
Merge

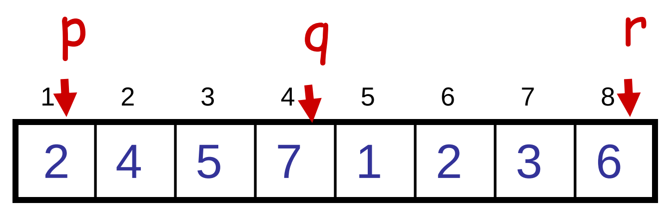


Merging

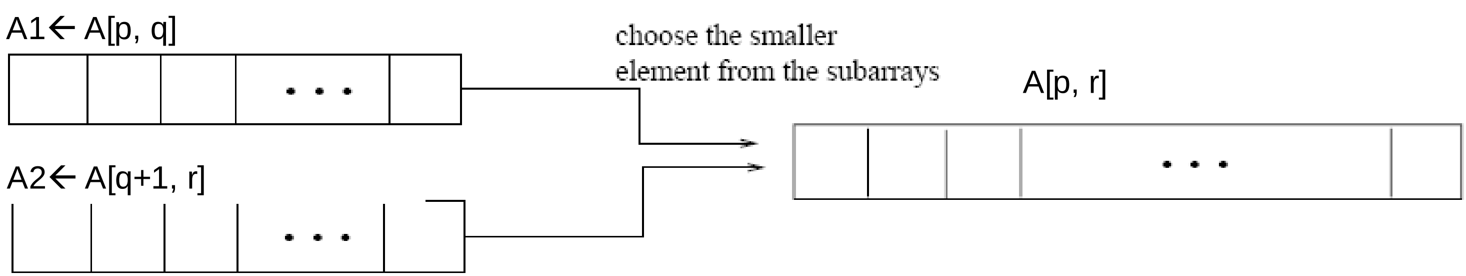
p			q				r
1	2	3	4	5	6	7	8
2	4	5	7	1	2	3	6

- **Input:** Array A and indices p, q, r such that $p \le q < r$
  - Subarrays $A[p \,..\, q]$ and $A[q + 1 \,..\, r]$ are sorted
- **Output:** One single sorted subarray $A[p \,..\, r]$

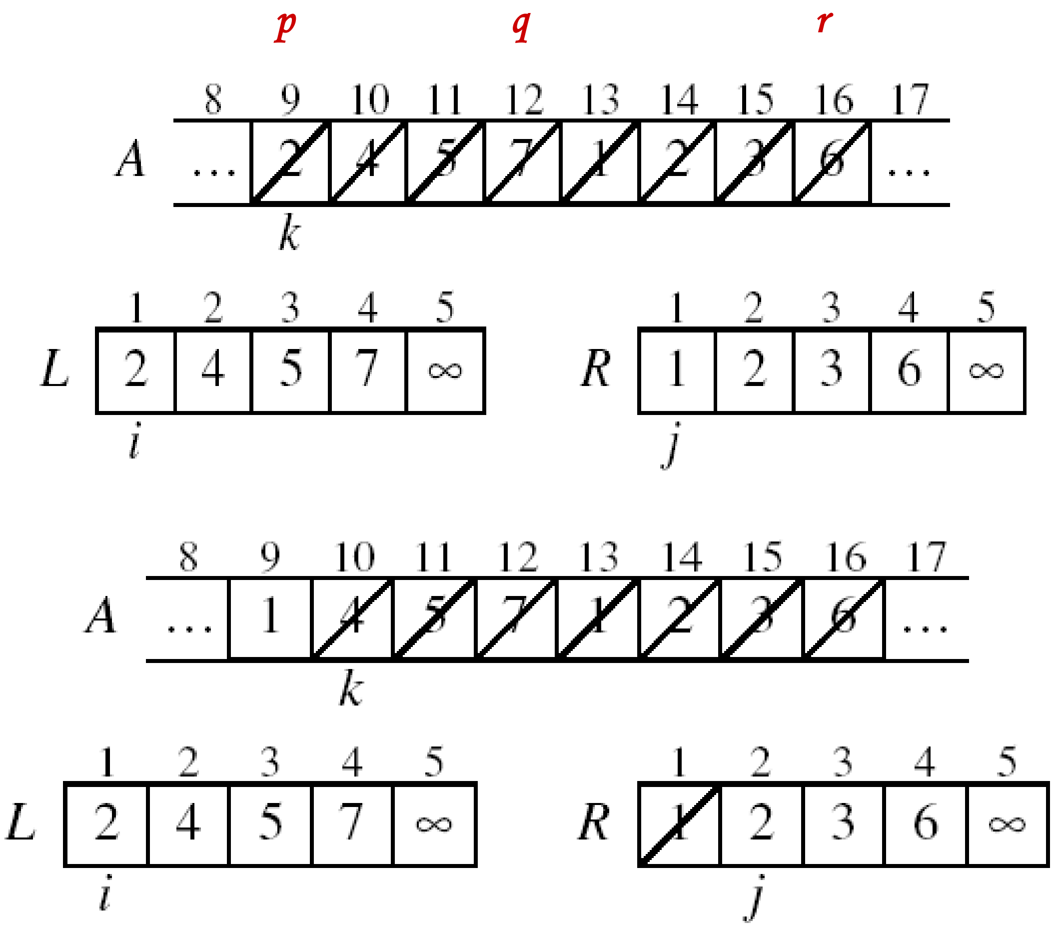
Merging



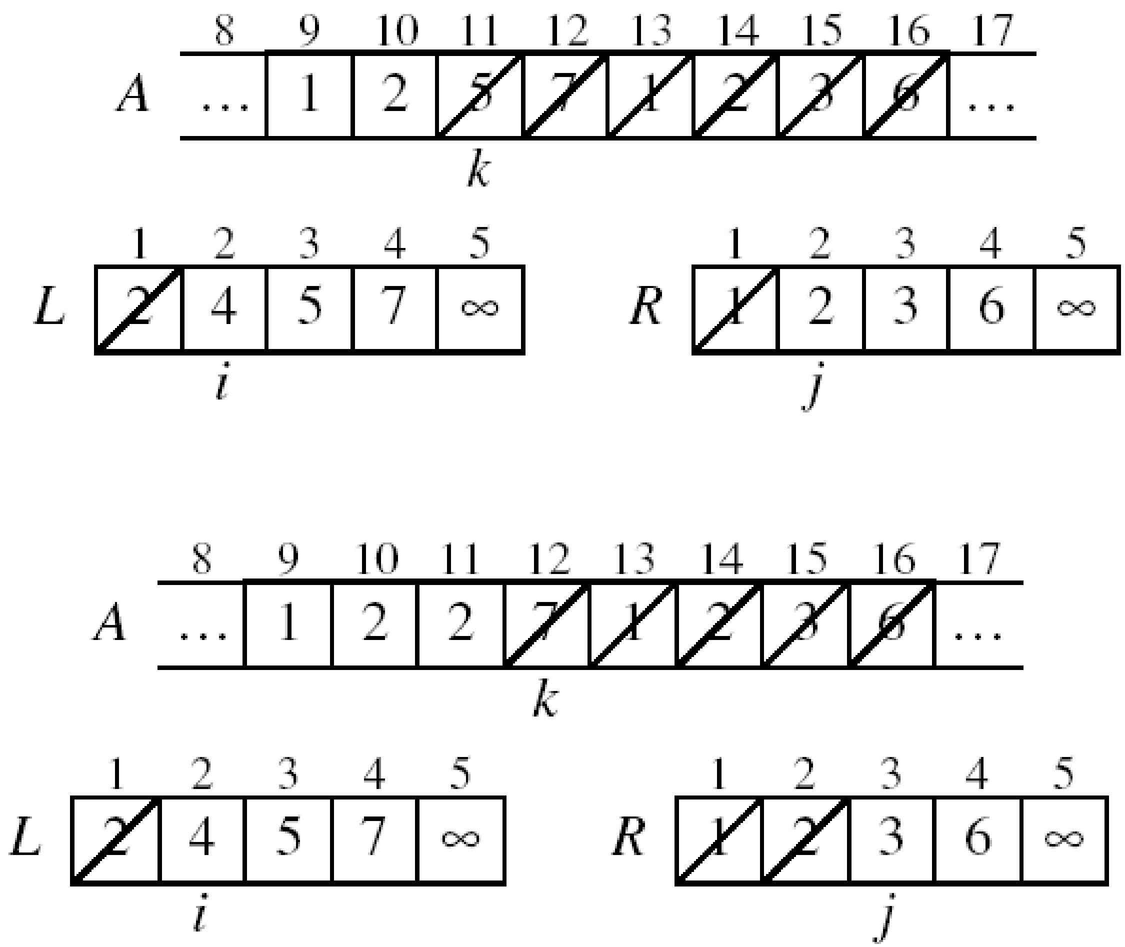
- Idea for merging:
  - Two piles of sorted cards
    - Choose the smaller of the two top cards
    - Remove it and place it in the output pile
  - Repeat the process until one pile is empty
  - Take the remaining input pile and place it face-down onto the output pile



Example: MERGE(A, 9, 12, 16)



Example: MERGE(A, 9, 12, 16)



Example (cont.)

	8	9	10	11	12	13	14	15	16	17
A	…	1	2	2	3	~~1~~	~~2~~	~~3~~	~~6~~	…
						k				

	1	2	3	4	5
L	~~2~~	4	5	7	∞
		i			

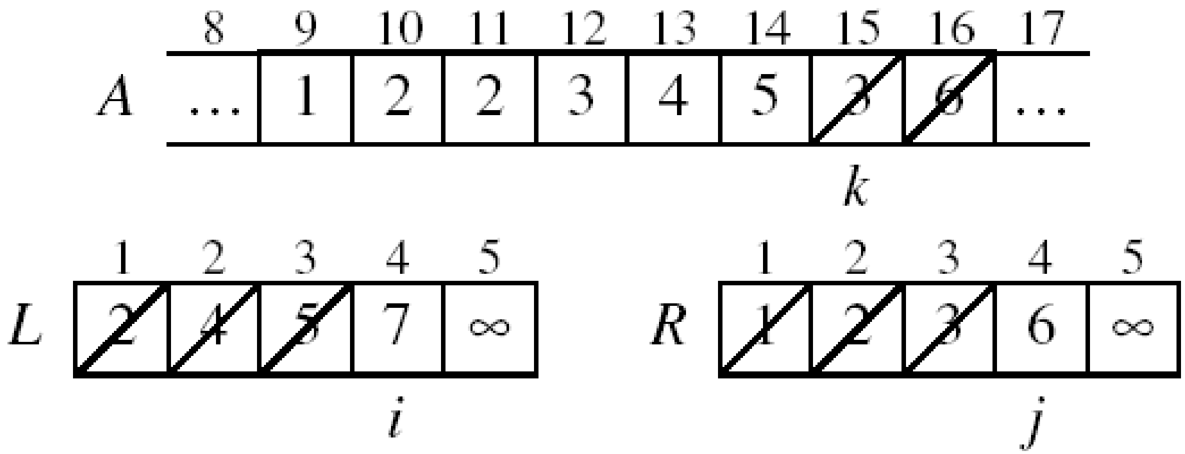
	1	2	3	4	5
R	~~1~~	~~2~~	~~3~~	6	∞
				j	

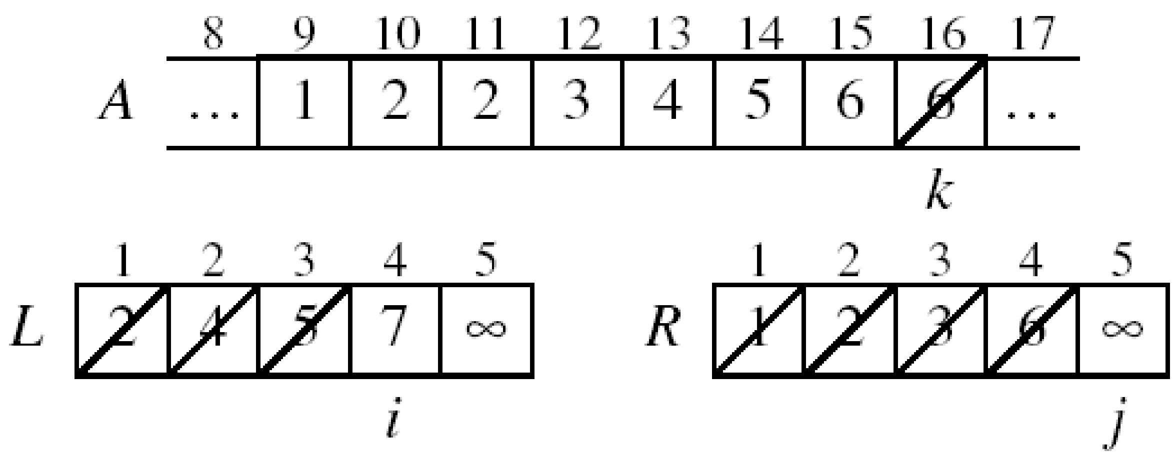
	8	9	10	11	12	13	14	15	16	17
A	…	1	2	2	3	4	~~2~~	~~3~~	~~6~~	…
							k			

	1	2	3	4	5
L	~~2~~	~~4~~	5	7	∞
			i		

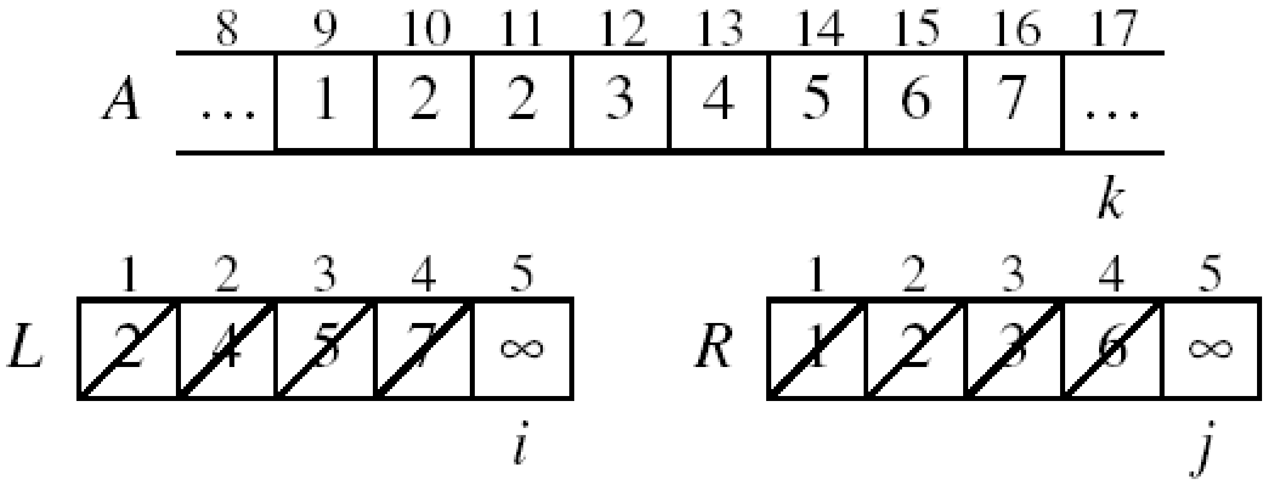
	1	2	3	4	5
R	~~1~~	~~2~~	~~3~~	6	∞
				j	

Example (cont.)





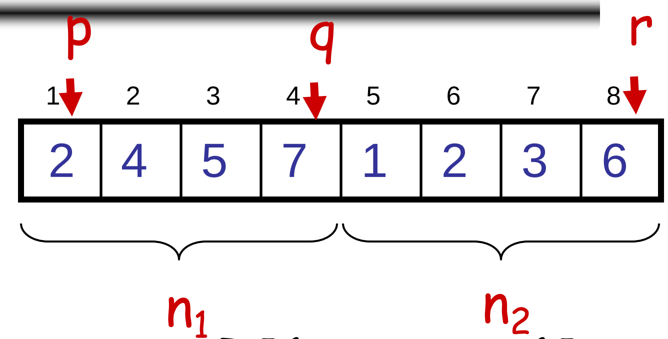
Example (cont.)



Done!

Merge - Pseudocode

Alg.: MERGE(A, p, q, r)



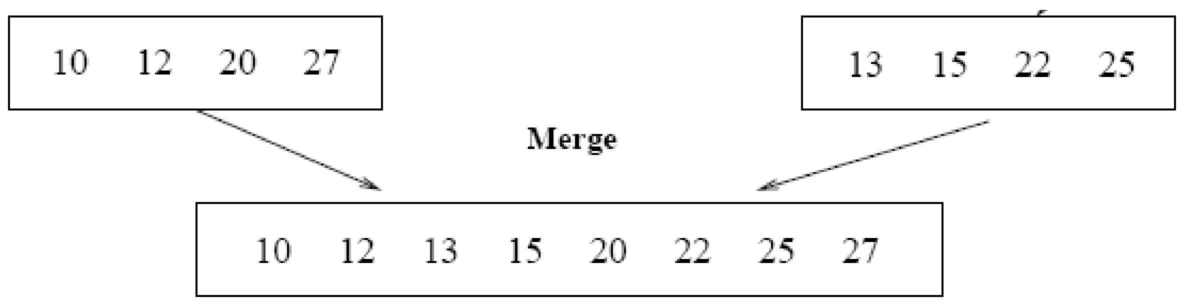
1. Compute n_1 and n_2
2. Copy the first n_1 elements into $L[1 \,.\,.\, n_1 + 1]$ and the next n_2 elements into $R[1 \,.\,.\, n_2 + 1]$
3. $L[n_1 + 1] \leftarrow \infty$; $R[n_2 + 1] \leftarrow \infty$
4. $i \leftarrow 1$; $j \leftarrow 1$
5. **for** $k \leftarrow p$ **to** r
6.   **do if** $L[\,i\,] \le R[\,j\,]$
7.   **then** $A[k] \leftarrow L[\,i\,]$
8. $i \leftarrow i + 1$
9.   **else** $A[k] \leftarrow R[\,j\,]$
10. $j \leftarrow j + 1$

	p			q	
L	2	4	5	7	∞

	q + 1			r	
R	1	2	3	6	∞

Running Time of Merge (assume last **for** loop)

- Initialization (copying into temporary arrays):
  - $\Theta(n_1 + n_2) = \Theta(n)$
- Adding the elements to the final array:
  - n iterations, each taking constant time $\Rightarrow \Theta(n)$
- Total time for Merge:
  - $\Theta(n)$



MERGE-SORT Running Time

- **Divide:**
  - compute q as the average of p and r: $D(n) = \Theta(1)$
- **Conquer:**
  - recursively solve 2 subproblems, each of size $n/2$
    $\Rightarrow 2T(n/2)$
- **Combine:**
  - MERGE on an n-element subarray takes $\Theta(n)$ time
    $\Rightarrow C(n) = \Theta(n)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + (n) & \text{if } n > 1 \end{cases}$$

Solve the Recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

Use Master's Theorem:

Compare n with $f(n) = cn$

Case 2: $T(n) = \Theta(n\lg n)$