Z-transform

~~Z-transform~~

Z-transform of sequence $f(k)$, denoted by $Z[f(k)]$ is given by

$$\boxed{Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}} = F(z)$$

(z is a complex no.)

- Linearity Property →

$$Z[af(k) + bg(k)] = aZ[f(k)] + bZ[g(k)]$$

- Change of scale property →

If $Z[f(k)] = F(z)$ then $Z[a^k f(k)] = F\left(\frac{z}{a}\right)$

and $Z[a^{-k} f(k)] = F(az)$

- Shifting property →

If $Z[f(k)] = F(z)$, then $Z[f(k \pm n)] = z^{\pm n} F(z)$

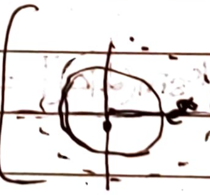
- Z-transform of standard functions →

① $Z[e^{k\alpha}] = \sum_{k=0}^{\infty} e^{\alpha k} z^{-k}$ (for $k \geq 0$)

$= 1 + \frac{e^{\alpha}}{z} + \frac{e^{2\alpha}}{z^2} + \frac{e^{3\alpha}}{z^3} + \dots$ (G.P.)

$= \frac{1}{1 - e^{\alpha}/z}$ for $\left|\frac{e^{\alpha}}{z}\right| < 1$ i.e. $e^{\alpha} < |z|$

$= \frac{z}{z - e^{\alpha}}$ for $|z| > e^{\alpha}$



[all points outside circle of radius e^{α} ($z = x + iy$)]

Q. Find $z[\sin(\alpha k)]$ and hence find $z[e^k \sin(\alpha k)]$.

Ans. $z[\sin(\alpha k)] = \sum_{k=0}^{\infty} \sin(\alpha k) z^{-k}$

$$= \sum_{k=0}^{\infty} \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \cdot z^{-k} = \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{ik\alpha} \cdot z^{-k} - \sum_{k=0}^{\infty} e^{-ik\alpha} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\left(1 + \frac{e^{i\alpha}}{z} + \frac{e^{2i\alpha}}{z^2} + \dots\right) - \left(1 + \frac{e^{-i\alpha}}{z} + \frac{e^{-2i\alpha}}{z} + \dots\right)\right]$$

$$= \frac{1}{2i}\left[\left(\frac{1}{1 - e^{i\alpha}/z}\right) - \left(\frac{1}{1 - e^{-i\alpha}/z}\right)\right] \text{ for } \left|\frac{e^{i\alpha}}{z}\right| < 1 \text{ and } \left|\frac{e^{-i\alpha}}{z}\right| < 1$$

$$= \frac{1}{2i}\left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}}\right] \text{ for } |e^{i\alpha}| < |z| \text{ and } |e^{-i\alpha}| < |z|$$

$$= \frac{1}{2i}\left[\frac{ze^{i\alpha} - ze^{-i\alpha}}{(z - e^{i\alpha})(z - e^{-i\alpha})}\right] \text{ for } |z| > 1$$

$$= \frac{1}{2i}\left[\frac{z(e^{i\alpha} - e^{-i\alpha})}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1}\right] \text{ for } |z| > 1$$

$$\therefore z[\sin(\alpha k)] = \frac{z \sin(\alpha)}{z^2 - 2z\cos(\alpha) + 1}$$

Using change of scale,

$$z[e^k \sin(\alpha k)] = F\left(\frac{z}{e}\right)$$

$$= \frac{\frac{z}{e}\sin(\alpha)}{(z/e)^2 - 2\frac{z}{e}\cos(\alpha) + 1}$$

$$\therefore z[e^k \sin(\alpha k)] = \frac{ez\sin(\alpha)}{z^2 - 2ez\cos(\alpha) + e^2}$$

Find : ① $z[a^{|k|}]$, ② $z[k^n] = -z\frac{d}{dz}z(k^{n-1})$

Q. $z[a^{|k|}]$

Ans. $= \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} = \sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$

$\therefore z[a^{|k|}] = \left(az + (az)^2 + (az)^3 + \dots\right) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots\right)$ $\left[\text{for } |az| < 1 \text{ and } \left|\frac{a}{z}\right| < 1\right]$

$= \frac{az}{1-az} + \frac{1}{1-a/z} = \frac{az}{1-az} + \frac{z}{z-a}$ $\left(\text{for } |z| < \frac{1}{a} \text{ and } |z| > a\right)$

$= \frac{-a^2z + z}{(1-az)(z-a)}$ $\left[\text{for } a < |z| < \frac{1}{a}\right.$ only possible when $\left.0 < a < 1\right]$

$\frac{1}{a} - 8$

Q. $z[k^n] = -z\frac{d}{dz}\left(z[k^{n-1}]\right)$

Ans. $\therefore z[k^{n-1}] = \sum_{k=0}^{\infty} k^{n-1} . z^{-k}$

$\therefore \frac{d}{dz} z[k^{n-1}] = \sum_{k=0}^{\infty} k^{n-1} . (-k) z^{-(k+1)}$

$\therefore -z\frac{d}{dz}\left(z[k^{n-1}]\right) = \sum_{k=0}^{\infty} k^n . z^{-k}$

$\therefore -z\frac{d}{dz}\left(z[k^{n-1}]\right) = z[k^n]$

Hence Proved.

Find ① $z[k]$ ② $z[k^2]$ ③ $z[k^2 \cdot a^k]$
④ $z[a^k \sin(k\theta)]$ ⑤ $z[k^2 \cdot e^{ak}]$ ⑥ $z\left[\frac{1}{k!}\right]$
⑦ $z\left[\frac{a^k}{k!}\right]$ ⑧ $z\left[\frac{1}{k}\right]$ $(k>0)$ ⑨ $z\left[\frac{1}{k(k+1)}\right]$

① Find out for $n=1$ from prev question.
② Find from ①
③ Change of scale on ②
④ ~~Take from~~ Same as question on pg. 136 (2nd part)
⑤ Change of scale on ②
⑥ We get $1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\dots \; (=e^{1/z})$, can solve then
⑦ Change of scale on ⑥
⑧ $z\left[\frac{1}{k}\right] = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} = \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\dots$

but $\log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \dots$

$\therefore \log(1-z) = -\left[z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\dots\right]$

$\therefore \log(1-1/z) = -\left[1/z + 1/2z^2 + 1/3z^3 + \dots\right]$

$\therefore z[1/k] = -\log\left(1-\frac{1}{z}\right) = \log\left(\frac{z}{z-1}\right)$

⑨ $z\left[\frac{1}{k(k+1)}\right] = z\left[\frac{1}{k}-\frac{1}{k+1}\right] = z[1/k] - z[1/(k+1)]$

$= \log\left(\frac{z}{z-1}\right) - \left(1+\frac{1}{2z}+\frac{1}{3z^2}+\frac{1}{4z^3}+\dots\right)$ for $\left|\frac{1}{z}\right|<1$

$\therefore z\left[\frac{1}{k(k+1)}\right] = \log\left(\frac{z}{z-1}\right) - z\left[\log\left(\frac{z}{z-1}\right)\right]$

$= (1-z)\log\left(\frac{z}{z-1}\right)$ for $|z|>1$

- Multiplication by n

If $z[f(k)] = F(z)$, then $z[k f(k)] = -z\frac{d}{dz}(F(z))$

In general,

$$z[k^n f(k)] = \left(-z\frac{d}{dz}\right)^n F(z)$$

Q. Find $Z[k^2 e^{ak}]$

Ans. $\therefore Z[k^2 e^{ak}] = \left(-z\frac{d}{dz}\right)^2 F(z)$ where $F(z) = Z[e^{ak}]$

$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}F(z)\right)$

$\therefore F(z) = Z[e^{ak}] = \dfrac{z}{z-e^a}$

$\therefore Z[k^2 e^{ak}] = \left(-z\frac{d}{dz}\right)\left(-z \times \dfrac{(z-e^a)(1) - z}{(z-e^a)^2}\right)$

$= \left(-z\frac{d}{dz}\right)\left(\dfrac{ze^a}{(z-e^a)^2}\right)$

$= -ze^a \times \left(\dfrac{(z-e^a)^2 - z \cdot 2(z-e^a)}{(z-e^a)^4}\right)$

$= -ze^a \left(\dfrac{z-e^a - 2z}{(z-e^a)^3}\right)$

$= \dfrac{ze^a(z+e^a)}{(z-e^a)^3}$ //

- Initial Value Theorem :

If $F(z) = Z[f(k)]$, then $f(0) = \lim_{z\to\infty} F(z)$

$f(1) = \lim_{z\to\infty} z[F(z) - f(0)]$

$f(2) = \lim_{z\to\infty} z^2\left[F(z) - f(0) - \frac{f(1)}{z}\right]$

$f(3) = \lim_{z\to\infty} z^3\left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2}\right]$

Q. If $F(z) = \frac{2z^2+5z+14}{(z-1)^4}$, find $f(2)$ and $f(3)$.

Ans. $F(z) = \frac{z^2\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4(1-1/z)^4}$

$\therefore f(0) = \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{2}{z^2} = 0$

$f(1) = \lim_{z\to\infty} z[F(z)-f(0)]$

$= z \times \frac{2}{z^2} = 0$

$f(2) = \lim_{z\to\infty} z^2\left[F(z)-f(0)-\frac{f(1)}{z}\right] = \lim_{z\to\infty} z^2 F(z)$

$\therefore \lim_{z\to\infty} z^2 \times \frac{2}{z^2} = 2$

$f(3) = \lim_{z\to\infty} z^3\left[F(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}\right]$

$\cancel{\lim_{z\to\infty}\left(\frac{2}{z^2}-\frac{2}{z^2}\right)}$ $= \lim_{z\to\infty} z\left[\frac{2+5/z+14/z^2}{(1-1/z)^4} - 2\right]$

$= \lim_{z\to\infty} \left[\frac{2z+5+14/z-2z+8-\frac{12}{z}+\frac{8}{z^2}-\frac{2}{z^3}}{(1-1/z)^4}\right]$

$\therefore f(3) = 5+8 = 13$

* Inverse Z-Transform

- Convolution Theorem

$$\boxed{\begin{array}{l}\text{If } Z^{-1}[F(z)] = f(n) \text{ and } Z^{-1}[G(z)] = g(n) \\ \text{then } Z^{-1}[F(z)\cdot G(z)] = \sum_{m=0}^{n} f(m)g(n-m)\end{array}}$$

Q. Using convolution, evaluate $z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

Ans. Let $F(z) = \dfrac{z}{z-a}$, $G(z) = \dfrac{z}{z-b}$

then $z^{-1}[F(z)] = a^n = f(n)$ - - - (from pg. 135 property)

and $z^{-1}[G(z)] = b^n = g(n)$

$\therefore z^{-1}[F(z)\,G(z)] = \sum_{m=0}^{n} f(m)\,g(n-m)$

$$= b^n \sum_{m=0}^{n} \left(\frac{a^m}{b^m}\right)$$

$\therefore z^{-1}[F(z)\,G(z)] = b^n\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots + \left(\frac{a}{b}\right)^n\right] \rightarrow$ [G.P. with ratio a/b]

$$= b^n \times \frac{1\left(\left(\frac{a}{b}\right)^{n}-1\right)}{\left(\frac{a}{b}-1\right)} \rightarrow \left(\text{using Sum} = \frac{a(r^n-1)}{r-1}\right) \quad (a>b)$$

$\therefore z^{-1}[F(z)\,G(z)] = \dfrac{a^{n+1} - b^{n+1}}{a-b}$ //

☆ Q. Using convolution, find $z^{-1}\left[\dfrac{z^3}{(z-a)^3}\right]$

then $z^{-1}[F(z)] = a^n = z^{-1}[G(z)]$, Considering $F(z) = \dfrac{z}{z-a} = G(z)$

$\therefore z^{-1}[F(z)\cdot G(z)] = \sum_{m=0}^{n} f(m)\,g(n-m)$

$$= a^n \sum_{m=0}^{n} 1 = (n+1)a^n \;//$$

Now, let $F(z) = \dfrac{z^2}{(z-a)^2}$, $G(z) = \dfrac{z}{z-a}$

$\therefore z^{-1}[F(z)] = (n+1)a^n = f(n)$, $z^{-1}[G(z)] = a^n = g(n)$

$\therefore z^{-1}[F(z)\cdot G(z)] = \sum_{m=0}^{n} f(m)\,g(n-m)$

$$= a^n \sum_{m=0}^{n} \frac{(m+1)a^m}{a^m} = a^n \sum_{m=0}^{n} (m+1)$$

$\therefore z^{-1}\left[\dfrac{z^3}{(z-a)^3}\right] = a^n\left[1+2+3+\dots+(n+1)\right]$

$$= a^n \frac{(n+1)(n+2)}{2} \;//$$

H.W.

Q. By convolution find $z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

Ans. $= z^{-1}\left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}\right] \to$ (where $a=\frac{1}{2}$, $b=\frac{1}{4}$)

- Binomial Expansion ↓

Q. Find $z^{-1}\left[\frac{4z}{z-a}\right]$ for (i) $|z|>|a|$ (ii) $|z|<|a|$

Ans. (i) $\frac{4z}{z-a} = 4\left(1-\frac{a}{z}\right)^{-1}$

$= 4\left[1+\frac{a}{z}+\frac{a^2}{z^2}+\cdots\right]$ for $\left|\frac{a}{z}\right|<1$

$= 4+4az^{-1}+4a^2z^{-2}+4a^3z^{-3}+\cdots$ for $|z|>|a|$

$= \sum_{k=0}^{\infty} 4a^k z^{-k} \to$ we know $z[f(k)] = \sum_{k=0}^{\infty} f(k)z^{-k} = F(z)$

$\therefore z^{-1}[F(z)] = f(k)$

$\therefore$ Here $f(k)=\{4a^k\}$ ← (notation for sequence)

(ii) $\frac{4z}{z-a} = -\frac{4z}{a}\left(1-\frac{z}{a}\right)^{-1}$

$= -\frac{4z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\cdots\right)$ for $\left|\frac{z}{a}\right|<1$

$= -\frac{4z}{a}-\frac{4z^2}{a^2}-\frac{4z^3}{a^3}-\cdots$ for $|z|<|a|$

$= -4\left[\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\cdots\right]$ for $|z|<|a|$

$= -4\sum_{k=1}^{\infty}\left(\frac{z}{a}\right)^k$

Let $k=-m$, $-4\sum_{m=-\infty}^{-1} z^{-m}a^{m}$

$\therefore z^{-1}\left[\frac{4z}{z-a}\right] = \{-4a^m\}_{m=-\infty}^{-1}$ for $|z|<|a|$

H.W.

Q. Find $Z^{-1}[(z-5)^{-3}]$ for $|z|>5$

$$\left\{ \text{Ans: } \begin{array}{ll} \frac{1}{2}(n-1)(n-2)5^{n-3}, & (n\geq 3) \\ 0, & (n<3) \end{array} \right.$$

Ans. $(1-z)^{-1} = 1+z+z^2+z^3+\cdots$

Diff. on both sides $\rightarrow (-1)(1-z)^{-2}(-1) = (1-z)^{-2} = 1+2z+3z^2+4z^3+\ldots$

Again diff. $\rightarrow (-2)(1-z)^{-3}(-1) = 2(1-z)^{-3} = 2+6z+12z^2+\ldots$

$\therefore (1-z)^{-3} = 1+3z+4z^2+\cdots$

- Partial fractions

Q. Find $z^{-1}\left[\dfrac{4z^2-2z}{z^3-5z^2+8z-4}\right]$

Ans. $= z^{-1}\left[\dfrac{2z(2z-1)}{(z-1)(z-2)^2}\right]$ $\therefore$ Let $\dfrac{2z-1}{(z-1)(z-2)^2} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-2)} + \dfrac{C}{(z-2)^2}$

$\therefore 2z-1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$

Put $z=1 \rightarrow 1 = A$ $\therefore A = 1$

Put $z=2 \rightarrow 3 = C$ $\therefore C=3$

Put $z=0 \rightarrow -1 = 4-3+2B$ $\therefore B=-1$

$\therefore \dfrac{(2z-1)(2z)}{(z-1)(z-2)^2} = 2z\left[\dfrac{1}{z-1} - \dfrac{1}{(z-2)} + \dfrac{3}{(z-2)^2}\right] = F(z)$

$\therefore z^{-1}[F(z)] = 2z^{-1}\left[\dfrac{z}{z-1}\right] - 2z^{-1}\left[\dfrac{z}{z-2}\right] + 6z^{-1}\left[\dfrac{z}{(z-2)^2}\right]$

we know, $z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$, also $z[na^n] = \dfrac{az}{(z-a)^2}$ $\therefore z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = \left\{\dfrac{1}{a} \times na^n\right\}$ (Remember!)

$\therefore z^{-1}[F(z)] = \left\{2(1^n) - 2\times(2^n) + \dfrac{6}{2}\times n\times 2^n\right\}$

$\therefore z^{-1}[F(z)] = \left\{2 - 2^{n+1} + 3n\times 2^n\right\}$

Q. Find Inverse Z-Transform of following functions!

① $\dfrac{z^3-20z}{(z-2)^3(z-4)}$ ② $\dfrac{z(z^2-5z+6.5)}{(z-2)(z-3)^2}$ for $2<|z|<3$

Ans ① $F(z)=\dfrac{z(z^2-20)}{(z-2)^3(z-4)}$ • Let $\dfrac{F(z)}{z}=\dfrac{A}{(z-2)}+\dfrac{B}{(z-2)^2}+\dfrac{C}{(z-2)^3}+\dfrac{D}{z-4}$

$\therefore z^2-20=A(z-2)^2(z-4)+B(z-2)(z-4)+C(z-4)+D(z-2)^3$

Put $z=2 \rightarrow -16=-2C \quad \therefore C=8$

Put $z=4 \rightarrow -4=8D \quad \therefore D=\cancel{-0.5}\ -1/2$

Put $z=0 \rightarrow -20=-16A+8B-32+4$

$\therefore -16A+8B=8$

Put $z=1 \rightarrow -19=-3A+3B-24+1/2$

$\therefore -3A+3B=9/2$

Should know ↓ $\boxed{Z[n^2.a^n]=\dfrac{az(z+a)}{(z-a)^3}}$

$-16A+8B=8$

$-8A+8B=12$

$-8A+0=-4 \quad \therefore A=+1/2, \quad B=2, \quad C=8, \quad D=-1/2$

↓

But C part cannot be solved then (we need an extra $(z+a)$ in numerator)

∴ Doing by another method ↓

$z^2-20=(A+Bz+Cz^2)(z-4)+D(z-2)^3$

Putting $z=2 \rightarrow$ ~~-16 = -2A - 4B - 8C~~

$-16=-2A-4B-8C$

Putting $z=4 \rightarrow -4=8D \quad \therefore D=-1/2$

⋮ Solving ahead, we get

$A=6, B=0, C=1/2, D=-1/2$

(continued)→

$$F(z) = z\left[\frac{6+z^2/2}{(z-2)^3} - \frac{1}{2(z-4)}\right]$$

$$= z\left[\frac{12+z^2}{2(z-2)^3} - \frac{1}{2(z-4)}\right]$$

$$= \frac{1}{2}\left[\frac{z(12+z^2)}{(z-2)^3} - \frac{z}{z-4}\right]$$

$$= \frac{1}{2}\left[\frac{z(z-2)^2 + 4z^2 + 8z}{(z-2)^3} - \frac{z}{z-4}\right]$$ → continued ↓

$$= \frac{1}{2}\left[\frac{z}{z-2} + \frac{4z(z+2)}{(z-2)^3} - \frac{z}{z-4}\right]$$

$$\therefore Z^{-1}[F(z)] = \frac{1}{2}\left(2^n + 2 \cdot n^2 . 2^n - 4^n\right)$$

$$= \frac{1}{2}\left(2^n + n^2 2^{n+1} - 4^n\right)$$ //

Ans ② $F(z) = \dfrac{2(z^2-5z+6.5)}{(z-2)(z-3)^2}$, $2 < |z| < 3$

$\therefore 2(z^2-5z+6.5) = A(z-3)^2 + B(z-2)(z-3) + C(z-2)$

Putting $z=2$, → $A=1$

$z=1$, → $C=1$

$z=0$, → $B=1$

$$\therefore F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2} \rightarrow \left|\frac{2}{z}\right| < 1 \text{ and } \left|\frac{z}{3}\right| < 1$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$\therefore F(z) = \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\dots\right) - \frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\dots\right) + \frac{1}{9}\left(1+\frac{2z}{3}+\frac{3z^2}{3^2}+\dots\right)$$

(continued) →

$$\therefore F(z) = \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} \cdot z^n + \frac{1}{9} \sum_{n=0}^{\infty} \frac{(n+1)}{3^n} \cdot z^n$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3^{n+1}} - \frac{(n+1)}{3^{n+2}} \right) \cdot z^n$$

Put $n = -k$,

$$\therefore F(z) = \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n} - \sum_{k=-\infty}^{0} \left(\frac{1}{3^{1-k}} - \frac{(1-k)}{3^{2-k}} \right) \cdot z^{-k}$$

$$\therefore Z^{-1}[F(z)] = \begin{cases} 2^{n-1}, & \text{for } n \geq 1 \\ -\left(\frac{1}{3^{1-k}} - \frac{(1-k)}{3^{2-k}} \right), & \text{for } -\infty \leq k \leq 0 \end{cases}$$

Q. Homework question from pg. 143.

Ans. $Z^{-1}[(z-5)^{-3}]$, for $|z| > 5$

$$(z-5)^{-3} = z^{-3}\left(1 - \frac{5}{z}\right)^{-3}$$

$$= \frac{1}{z^3}\left[1 + 3 \times \left(\frac{5}{z}\right) + 6 \times \left(\frac{5}{z}\right)^2 + \cdots\right]$$

$$= \frac{1}{2z^3}\left[(1 \times 2) + (2 \times 3)\left(\frac{5}{z}\right) + (3 \times 4)\left(\frac{5}{z}\right)^2 + (4 \times 5)\left(\frac{5}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{2z^3} \sum_{n=0}^{\infty} (n+1)(n+2)\left(\frac{5}{z}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2)\, 5^{n} \cdot z^{-(n+3)}$$

(continued) →

Put $n+3=k$,

$$\therefore \frac{1}{2}\sum_{k=3}^{\infty}(k-2)(k-1)\cdot 5^{k-3}\cdot z^{-k}$$

$$\therefore z^{-1}[F(z)] = \left\{\frac{(k-1)(k-2)5^{k-3}}{2}\right\}, \text{ for } k \geq 3$$