Vector Integration

($\oint \rightarrow$ closed contour)

- Line Integral :

Let $\bar{F}$ be a vector field in the region R then let C be any curve in this region, let $\bar{r}$ be position vector of the point P on the curve C.

Then line integral of $\bar{F}$ along the curve C is given by:

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = \int_C F_1 dx + F_2 dy + F_3 dz}$$

A vector field $\bar{F}$ is called conservative if there exists a scalar potential ϕ such that $\bar{F} = \nabla \phi$.

☆ [Important Note: $\bar{F}$ is conservative iff $\bar{F}$ is irrotational.]

Let $\bar{F}$ be a continuous vector field. Then, following statements are equivalent:

1. $\int_C \bar{F} \cdot d\bar{r}$ is independent of the path joining the endpoints, and it only depends upon the endpoints.

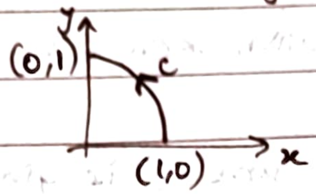
☆

2. $\bar{F}$ is a conservative field (i.e. $\bar{F} = \nabla \phi$).

3. For any closed curve C, $\oint_C \bar{F} \cdot d\bar{r} = \text{work done} = 0$

Q. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = \cos y\,\hat{i} - x\sin y\,\hat{j}$ and c is the curve $y = \sqrt{1-x^2}$ in xy-plane, from (1,0) to (0,1).

Ans. $y = \sqrt{1-x^2} \rightarrow \therefore y^2 = 1 - x^2 \quad y^2 + x^2 = 1$ (unit circle)



$$\int_C \bar{F} \cdot d\bar{r} = \int_C \cos y\, dx - x\sin y\, dy$$

(But $\int \cos y\, dx = x\cos y$
$-\int x\sin y\, dy = x\cos y$)

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \int_C d(x\cos y) = [x\cos y]_{(1,0)}^{(0,1)}$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = 0 - 1 = -1$$

Q. Find work done in moving a particle once around the circle $x^2+y^2=a^2$, $z=0$ (xy-plane) in the force field given by:
$\bar{F} = \sin y\,\hat{i} + (x + x\cos y)\,\hat{j}$

Ans. [closed path for C.]

$$\text{work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_C \sin y\,dx + (x + x\cos y)\,dy$$

$$= \int_C (\sin y\,dx + x\cos y\,dy) + x\,dy$$

$$= \int_C d(x\sin y) + \int_C x\,dy$$

Let $\int_C \sin y\,dx + x\cos y\,dy = \int_C \bar{F_1}\cdot d\bar{r}$

$$= \int d\phi$$

$\bar{F_1}\cdot d\bar{r} = d\phi = \nabla\phi\cdot d\bar{r}$

$\therefore \bar{F_1} = \nabla\phi \rightarrow$ conservative

$\therefore \int_C \bar{F_1}\cdot d\bar{r} = 0$

$$\therefore \text{workdone} = 0 + \int_C x\,dy$$

$$= \oint_C x\,dy$$

(Put $x = a\cos\theta$, $y = a\sin\theta$)

$$\therefore \text{workdone} = \int_{\theta=0}^{2\pi} a\cos\theta \cdot a\cos\theta\,d\theta =$$

$$= a^2 \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$\text{work done} = \frac{a^2}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{2\pi} = \pi a^2$$

Q. Prove that $\bar{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k}$ is conservative.

Find ① Scalar potential for $\bar{F}$.

② work done in moving an object from $(0,1,-1)$ to $(\pi/2,-1,2)$

Ans. Conservative if and only if irrotational.

$\therefore \nabla \times \bar{F} = \bar{0}$ ← Prove

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\therefore (0-0)\hat{i} - \hat{j}(3z^2 - 3z^2) + \hat{k}(2y\cos x - 2y\cos x) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

LHS = RHS

Hence Proved.

① Scalar potential ϕ, such that $\bar{F} = \nabla\phi$

$$\therefore \frac{\partial\phi}{\partial x} = y^2\cos x + z^3, \quad \frac{\partial\phi}{\partial y} = 2y\sin x - 4, \quad \frac{\partial\phi}{\partial z} = 3xz^2+2$$

$$\therefore d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2+2)dz$$

$$\cancel{= (y^2\cos x + 2y\sin x)}$$

$$= (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) - 4dy + 2dz$$

$$d\phi = d(y^2\sin x) + d(xz^3) - d(4y) + d(2z)$$

$$\cancel{\therefore \phi = y^2\sin x}$$

$$d\phi = d(y^2\sin x + xz^3 - 4y + 2z)$$

$$\therefore \phi = y^2\sin x + xz^3 - 4y + 2z + C \quad \text{(scalar potential)}$$

② work done $= \int_C \bar{F}\cdot d\bar{r}$

But $\bar{F}\cdot d\bar{r} = F_1dx + F_2dy + F_3dz = d\phi$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \int d\phi = [\phi]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$\therefore \text{work done} = (1 + 4\pi + 4 + 4 + c) - (0 + 0 - 4 - 2 + c) = 9 + 4\pi + 6$$

$$= 4\pi + 15$$

- Green's Theorem

→ Relation between line integral and area Integral.

Let $\bar{F} = P\hat{i} + Q\hat{j}$

Let $\bar{F}$ be a continuous vector field in the region R.

$\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are also continuous in the region R.

Let c be positively-oriented boundary of the region R. (anti-clockwise)

then

$$\oint_c P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

→ Vector form of Green's Theorem

$$\int_c \bar{F}\cdot d\bar{r} = \iint_S \hat{N}\cdot(\nabla\times\bar{F})\,ds \qquad \bar{F} = P\hat{i} + Q\hat{j}$$

where $\hat{N}$ is unit normal vector along z-axis.

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \hat{i}\left(0 - \frac{\partial Q}{\partial z}\right) - \hat{j}\left(0 - \frac{\partial P}{\partial z}\right) + \hat{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$\therefore \hat{N}\cdot(\nabla\times\bar{F}) = \hat{k}\cdot(\nabla\times\bar{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Q. Evaluate using Green's Theorem –
$\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is boundary of region bounded by ① $y = \sqrt{x}$ and $y = x$
② $y = \sqrt{x}$ and $y = x^2$

Ans. $P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$\frac{\partial Q}{\partial x} = -6y$, $\frac{\partial P}{\partial y} = -16y$

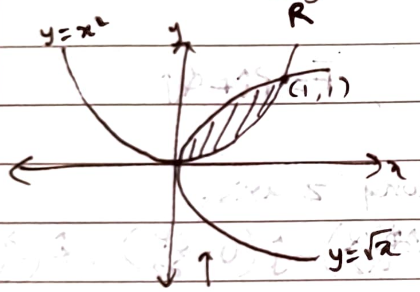
$\therefore \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$

By Green's Theorem,

$$\int_c \bar{F} \cdot d\bar{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_R (10y)\, dx\, dy$$

②



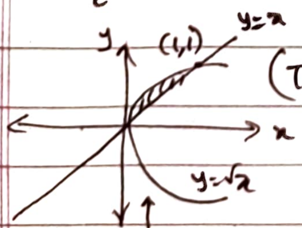
← } Taking vertically first (y)

$$\therefore \int_c \bar{F} \cdot d\bar{r} = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\, dy\, dx$$

$$= 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 5 \int_{x=0}^{1} (x - x^4)\, dx$$

$$\therefore \int_c \bar{F} \cdot d\bar{r} = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left(\frac{3}{10}\right) = \frac{3}{2}$$

①



(Taking vertically first)

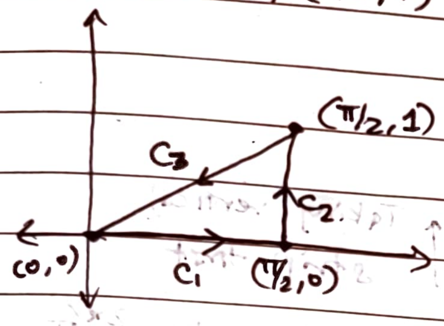
$$\therefore \int_c \bar{F} \cdot d\bar{r} = 10 \int_{x=0}^{1} \int_{y=x}^{\sqrt{x}} y\, dy\, dx$$

$$= 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x}^{\sqrt{x}} dx = 5 \int_{x=0}^{1} (x - x^2)\, dx$$

$$\therefore \int_c \bar{F} \cdot d\bar{r} = 5\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 5\left(\frac{1}{6}\right) = 5/6$$

Q. Verify Green's Theorem for $\oint_c (y-\sin x)dx + \cos x\,dy$ where c is boundary of a ΔOAB, whose vertices are $(0,0)$, $(\pi/2, 0)$, $(\pi/2, 1)$

Ans.



(In verifying, need to check both LHS and RHS of Green's Theorem, and prove they are equal)

Along C_1: $y=0$, $dy=0$

$$\therefore \int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^{\pi/2} -\sin x\,dx + 0 \qquad \left(\int \bar{F}\cdot d\bar{r} = \oint (y-\sin x)dx + \cos x\,dy\right)$$

$$= [\cos x]_0^{\pi/2} = -1$$

Along C_2: $x = \pi/2$, $dx = 0$

$$\therefore \int_{C_2} \bar{F}\cdot d\bar{r} = \int 0 = 0$$

Along C_3: $x = \frac{\pi}{2}y$, $dx = \pi/2\,dy$ $\left(y = \frac{2x}{\pi},\ dy = \frac{2}{\pi}dx\right)$

$$\therefore \int_{C_3} \bar{F}\cdot d\bar{r} = \int_{x=\pi/2}^{0} \left(\frac{2x}{\pi} - \sin x\right)dx + \cos x\left(\frac{2}{\pi}dx\right)$$

$$= \left[\frac{x^2}{\pi} + \cos x + \frac{2\sin x}{\pi}\right]_{\pi/2}^{0}$$

$$= 1 - \left(\pi/4 + \frac{2}{\pi}\right) = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r} + \int_{C_3} \bar{F}\cdot d\bar{r}$$

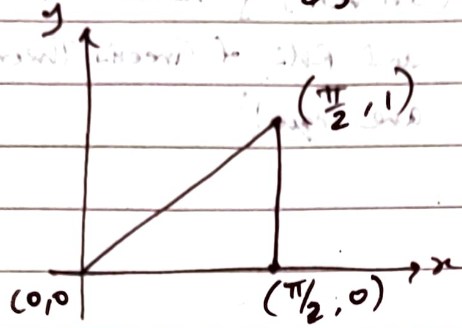
$$= -1 + 0 + 1 - \pi/4 - 2/\pi$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = LHS$$

For RHS,

$P = y - \sin x$, $Q = \cos x$

$\therefore \frac{\partial Q}{\partial x} = -\sin x$, $\frac{\partial P}{\partial y} = 1$



↑ Taking vertical strip first

$y \to 0$ to $2x/\pi$

$x \to 0$ to $\pi/2$

We need $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\, dy\, dx$$

$$= \int_{x=0}^{\pi/2} \left[-y \sin x - y\right]_0^{2x/\pi} dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} (x \sin x + x)\, dx$$

$$= \frac{-2}{\pi} \left[x(x - \cos x) - (1)\left(\frac{x^2}{2} - \sin x\right)\right]_0^{\pi/2}$$

$$= \frac{-2}{\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{8} + 1 - 0\right]$$

$$= \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = \frac{-\pi}{4} - \frac{2}{\pi} \quad = \text{RHS}$$

LHS = RHS, Hence verified Green's Theorem.

- Stokes' Theorem

Let $\bar{F}$ be a continuous vector field, then

$$\boxed{\oint_C \bar{F}\cdot d\bar{r} = \iint_S \hat{N}\cdot(\nabla\times\bar{F})\,ds}$$

(should be open surface)

where $\hat{N}$ is unit outward normal vector to an element ds (of S)

(Green's Theorem ~~(vector form)~~ (vector form) is special case of Stokes' Theorem)

Important Note:

If S and S_1 are 2 surfaces having the same boundary C, then $\iint_S \hat{N}\cdot(\nabla\times\bar{F})\,ds = \iint_{S_1}\hat{N}\cdot(\nabla\times\bar{F})\,ds_1 = \oint_C \bar{F}\cdot d\bar{r}$

(E.g. )

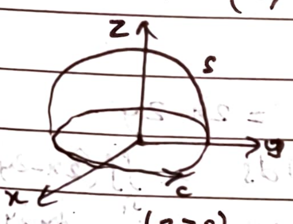
Q. Evaluate $\oint_C \bar{F}\cdot d\bar{r}$ using Stokes' Theorem, where $\bar{F} = y\hat{i}+z\hat{j}+x\hat{k}$ over the surface $S: x^2+y^2 = 1-z$, $z\geq 0$

Ans. $\nabla\times\bar{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & z & x\end{vmatrix} = -\hat{i}-\hat{j}-\hat{k}$

$x^2+y^2 = 1-z \rightarrow \therefore x^2+y^2 = -(z-1)$

$X^2+Y^2 = -Z$ (Putting X, Y, Z = 0, and finding x, y, z gives vertex)

$\therefore$ vertex = (0, 0, 1)



$\therefore$ Consider $S_1: x^2+y^2 \leq 1$ (has same boundary)

Here, $\hat{N} = \hat{k}$

$\therefore \hat{N}\cdot(\nabla\times\bar{F}) = -1$

$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_{S_1}(-1)\,dx\,dy = -\iint_{S_1} dx\,dy$

(area of circle in this case)

$\therefore \oint_C \bar{F}\cdot d\bar{r} = -\pi$

Q. Apply Stokes' theorem to evaluate $\int_C \bar{F} \cdot d\bar{r}$ for $\bar{F} = (y^2+z^2-x^2)\hat{i} + (z^2+x^2-y^2)\hat{j} + (x^2+y^2-z^2)\hat{k}$ over the surface $x^2+y^2-2ax+az=0$ above plane $z=0$.

Ans. $\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$

$= (2y-2z)\hat{i} - (2x-2z)\hat{j} + (2x-2y)\hat{k}$

Now,

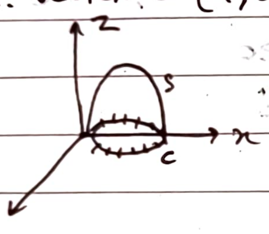
$(x^2-2ax)+y^2 = -az$

$\therefore (x-a)^2+y^2 = -a(z-a)$

$\downarrow \quad \downarrow \quad \downarrow$

$X^2+Y^2 = -aZ$ (paraboloid) [lower half]

$\therefore$ vertex $= (a,0,a)$

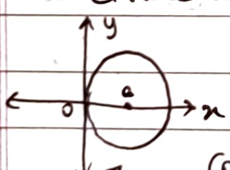


Put $z=0$ in original surface equation,

$x^2+y^2-2ax=0$

$\therefore (x-a)^2+y^2=a^2$

$\Rightarrow$ circle with centre $=(a,0)$ and $r=a$



Here $\hat{N} = \hat{k}$ $\therefore \hat{N}\cdot(\nabla\times\bar{F}) = 2x-2y$

$\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_{S_1} \hat{N}\cdot(\nabla\times\bar{F})\,ds = \iint_{S_1}(2x-2y)\,dx\,dy$

Converting to polar $\rightarrow$ $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$\therefore 2\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2a\cos\theta} r(\cos\theta-\sin\theta)\,r\,dr\,d\theta$ [limits found out like last sem]

$\therefore \frac{2}{3}\int_{-\pi/2}^{\pi/2} 8a^3\cos^3\theta(\cos\theta-\sin\theta)\,d\theta = \frac{16a^3}{3}\times 2\int_0^{\pi/2}\cos^4\theta\,d\theta$ [$\sin\theta\cdot\cos^3\theta = 0$]

$= \frac{32a^3}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = 2\pi a^3$

$\uparrow$

[REDUCTION FORMULA]

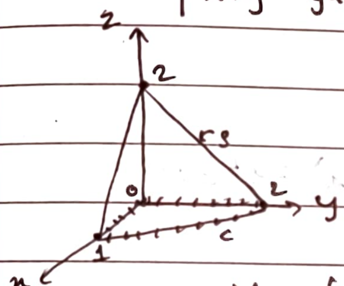
NOTE: Reduction Formula:

$$\int_0^{\pi/2} \cos^n\theta\, d\theta = \int_0^{\pi/2} \sin^n\theta\, d\theta$$

$$= \begin{cases} \frac{(n-1)}{(n)} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-3)} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n = \text{even} \\ \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-3)} \cdots \frac{2}{3} & \text{if } n = \text{odd} \end{cases}$$

Q. Using Stokes' Theorem, find ~~integral~~ $\int_C \bar{F} \circ d\bar{r}$ for $\bar{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$ and s is surface of plane $2x+y+z=2$ in the first octant.

Ans. $\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & y+z & -x \end{vmatrix} = -\hat{i} + \hat{j} - \hat{k}$



Putting x, y, and z 0 in pairs, we find intercepts on axes.

Consider $\phi = 2x+y+z-2$

$\therefore$ Normal to $\phi = \nabla\phi = 2\hat{i} + \hat{j} + \hat{k}$

$\therefore \hat{N}$ = unit vector in dir. of $\nabla\phi = (2\hat{i} + \hat{j} + \hat{k})/\sqrt{6}$

$\therefore \hat{N} \circ (\nabla \times \bar{F}) = \frac{-2}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = -2/\sqrt{6}$

[Now, $ds = \frac{dx\,dy}{|\hat{N} \circ \hat{k}|}$ (xy plane projection) Similarly $ds = \frac{dy\,dz}{\hat{N} \circ \hat{i}}$ and ...
$\rightarrow$ (only +ve value)]

$\therefore ds = \frac{dx\,dy}{1/\sqrt{6}} = \sqrt{6} \cdot dx\,dy$

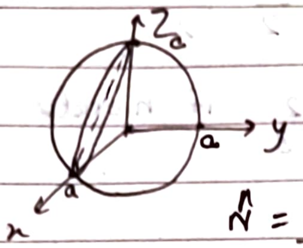
$\therefore \int_C \bar{F} \circ d\bar{r} = \iint_S \hat{N} \circ (\nabla \times \bar{F})\, ds = \iint_S -\frac{2}{\sqrt{6}}\, dx\,dy\,\sqrt{6}$

$= -\iint 2\,dx\,dy$ (It is a triangle on xy plane)

$= -2 \times \left(\frac{1}{2} \times 2 \times 1\right) = -2$ [or can do by normal method]

Q. Apply Stokes' Theorem to evaluate $\int_C y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ and $x+z = a$

Ans.



Let $\phi = x+z-a$

$\therefore \nabla\phi = \hat{i}+\hat{k}$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}, \quad \nabla\times\bar{F} = -\hat{i}-\hat{j}-\hat{k}$$

$$\therefore \hat{N}\cdot(\nabla\times\bar{F}) = -\sqrt{2}$$

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|} \text{ (xy projection) } = \sqrt{2}\,dx\,dy$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \iint_S \hat{N}\cdot(\nabla\times\bar{F})\,ds$$

$$= -2\iint_S dx\,dy$$

Solving $x^2+y^2+z^2 = a^2$ and $x+z = a$

$$\therefore x^2+y^2+(a-x)^2 = a^2, \quad 2(x^2-ax)+y^2 = 0$$

$$2\left(x^2-ax+\frac{a^2}{4}\right)+y^2 = \frac{a^2}{2}$$

$$\therefore \left(x-\frac{a}{2}\right)^2 + \frac{y^2}{2} = \frac{a^2}{4}$$

$$\therefore \frac{(x-a/2)^2}{a^2/4} + \frac{y^2}{a^2/2} = 1$$

$$= \frac{(x-a/2)^2}{(a/2)^2} + \frac{y^2}{(a/\sqrt{2})^2} = 1$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = -2\times(\text{area of ellipse})$$

$$= -2\times(\pi ab)$$

$$\therefore -2\times\left(\pi\times\frac{a}{2}\times\frac{a}{\sqrt{2}}\right)$$

$$= -\frac{\pi a^2}{\sqrt{2}}$$

H.W.

Q. Apply Stokes Theorem to evaluate $\oint 3y\,dx + 4z\,dy + 6y\,dz$

• **Gauss' Divergence Theorem** ↓

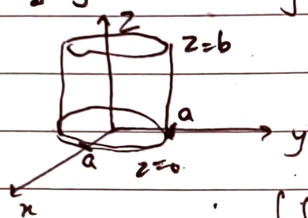
Let $\bar{F}$ be a continuous vector field, then

$$\boxed{\iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dv}$$

(Region should be closed region)

Q. Use divergence theorem for $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over cylindrical region $x^2+y^2=a^2$, $(z=0,\ z=b)$

Ans.



$$\nabla\cdot\bar{F} = 4-4y+2z$$

$$\therefore \iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV$$

$$= \iiint_V (4-4y+2z)\,dx\,dy\,dz$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ → $dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\cancel{\therefore \iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{b}(4-4y+2z)\,r\,dr\,d\theta\,dz}$$

$$\therefore \iint_S \hat{N}\cdot\bar{F}\,ds = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{b}(4-4r\sin\theta+2z)\,r\,dr\,d\theta\,dz$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\left[(4-4r\sin\theta)b+b^2\right]r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(2ba^2 - 4b\sin\theta\,\frac{a^3}{3} + \frac{a^2b^2}{2}\right)d\theta$$

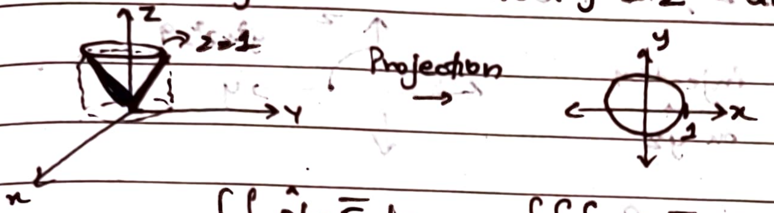
$$= 2a^2b(2\pi) + \frac{a^2b^2}{2}\times 2\pi$$

$\cancel{=d}$

$$= 4\pi a^2 b + \pi a^2 b^2$$ //

Q. Use divergence theorem to find $\iint_S \hat{N}\cdot\bar{F}\,ds$ where $\bar{F} = x\hat{i} + y\hat{j} + z^2\hat{k}$ and s is the closed surface bounded by the cone $x^2+y^2=z^2$ and $z=1$

Ans.



$$\iint_S \hat{N}\cdot\bar{F}\,ds = \iiint_V \nabla\cdot\bar{F}\,dV$$

$$= \iiint_V (1+1+2z)\,dx\,dy\,dz$$

Use cylindrical coordinates $\rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$, $z=z$, $dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\therefore \iint_S \hat{N}\cdot\bar{F}\,ds = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{1} (2+2z)\,r\,dr\,d\theta\,dz$$

(Revise how to select limits)

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left[2z+z^2\right]_r^1 r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (3r - 2r^2 - r^3)\,dr\,d\theta$$

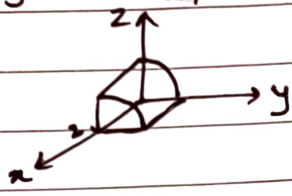
$$= \int_{\theta=0}^{2\pi} \left[\frac{3r^2}{2} - \frac{2r^3}{3} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{7}{12}\int_{\theta=0}^{2\pi} d\theta = \frac{7}{12}\times 2\pi$$

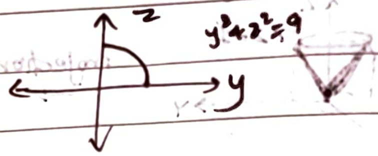
$$\therefore \iint_S \hat{N}\cdot\bar{F}\,ds = \frac{7\pi}{6}$$

Q. Evaluate $\iint_S \hat{N} \cdot \bar{F} \, ds$ using divergence theorem, for $\bar{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ taken over region bounded by $y^2+z^2=9$, and $x=2$, in the first octant.

Ans.



Projection on yz



$$\nabla \cdot \bar{F} = 4xy - 2y + 8xz$$

$$\iint_S \hat{N} \cdot \bar{F} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$$

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} \int_{x=0}^{2} (4xy - 2y + 8xz) \, dx \, dy \, dz$$

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} \left[2x^2y - 2xy + 4x^2z\right]_0^2 \, dx \, dy \, dz$$

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z) \, dy \, dz$$

$$= \int_{y=0}^{3} \left[4yz + 8z^2\right]_0^{\sqrt{9-y^2}} dy$$

$$= \int_{y=0}^{3} \left(4y\sqrt{9-y^2} + 8(9-y^2)\right) dy$$

~~Let~~ y $= \int_{y=0}^{3} -2\left(-2y\sqrt{9-y^2}\right) + 8(9-y^2) \, dy$

$$= \left[-2\frac{(9-y^2)^{3/2}}{3/2} + 8\left(9y - \frac{y^3}{3}\right)\right]_0^3$$

$$= \frac{-4}{3}\left(-9^{3/2}\right) + 8(27-9)$$

$$= \frac{4}{3} \times 27 + 144$$

$$\therefore \iint_S \hat{N} \cdot \bar{F} \, ds = 180$$