Vector Differentiation

- **Gradient:**

The gradient vector of $f(x,y)$ at point $P(x_0,y_0)$,

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}}$$

- **Directional Derivative:**

Directional derivative of function f in direction of unit vector $\hat{u}$, at point (a,b),

$$\boxed{D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u}}$$

→ Properties of Directional Derivative:

① $D_{\hat{u}} f = \nabla f \cdot \hat{u}$

$= |\nabla f|\,|\hat{u}| \cos\theta \qquad (|\hat{u}| = 1)$

$= |\nabla f| \cos\theta$

② $D_{\hat{u}} f$ is maximum if $\cos\theta = 1$ (i.e. $\theta = 0°$)

$\therefore D_u f = |\nabla f|$ (max value)

Function f increases most rapidly in direction of ∇f (or $\hat{u}$).

$D_u f$ is minimum if $\cos\theta = -1$ (i.e. $\theta = \pi$)

$\therefore D_u f = -|\nabla f|$ (min value)

Function f decreases most rapidly in direction of $-\nabla f$ (or $\hat{u}$).

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Q. ~~Gradient of f = $\nabla f = \frac{\partial f}{\partial x}\hat{i}$~~

→ $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(dx, dy, dz)$

$$\boxed{df = \nabla f \cdot d\bar{r}}$$

Q. Find $\phi(r)$ such that $\nabla\phi = -\frac{\bar{r}}{r^5}$, and $\phi(1) = 0$.

Ans. $\nabla\phi = -\frac{\bar{r}}{r^5} = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{5/2}}$

$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\frac{\partial\phi}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{5/2}}$, $\frac{\partial\phi}{\partial y} = \frac{-y}{(x^2+y^2+z^2)^{5/2}}$, $\frac{\partial\phi}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{5/2}}$

Now, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$= \frac{-x\,dx}{(x^2+y^2+z^2)^{5/2}} - \frac{y\,dy}{(x^2+y^2+z^2)^{5/2}} - \frac{z\,dz}{(x^2+y^2+z^2)^{5/2}}$

$= \frac{-1}{(x^2+y^2+z^2)^{5/2}}[x\,dx + y\,dy + z\,dz]$

Let $x^2+y^2+z^2 = t$ $\therefore 2x\,dx + 2y\,dy + 2z\,dz = dt$ $\therefore x\,dx + y\,dy + z\,dz = \frac{dt}{2}$

$\therefore d\phi = \frac{-dt}{2\cdot t^{5/2}}$ $\int d\phi = \frac{+1}{2}\times\left(\frac{+2}{3}\right)\cdot\frac{1}{t^{3/2}} + c$

$\therefore \phi = \frac{1}{3t^{3/2}} + c = \frac{1}{3(x^2+y^2+z^2)^{3/2}} + c = \frac{1}{3r^3} + c$

$\phi(1) = 0 = \frac{1}{3(1)} + c$ $\therefore c = -\frac{1}{3}$ $\therefore \phi(r) = \frac{1}{3r^3} - \frac{1}{3}$ //

Q. Prove that $\nabla f(r) = f'(r) \cdot \frac{\bar{r}}{r}$. Hence find f if $\nabla f = 2r^4 \bar{r}$.

Ans. $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

By chain rule, $\frac{\partial f}{\partial x} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}$. (similarly for y and z)

$\therefore \nabla f = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}\hat{i} + \frac{df}{dr} \cdot \frac{\partial r}{\partial y}\hat{j} + \frac{df}{dr} \cdot \frac{\partial r}{\partial z}\hat{k}$

But $r^2 = x^2 + y^2 + z^2$

$\therefore 2r \frac{\partial r}{\partial x} = \cancel{2x}\, 2x$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$ ← Remember!

$\therefore \nabla f = \frac{df}{dr} \cdot \frac{x}{r}\hat{i} + \frac{df}{dr} \cdot \frac{y}{r}\hat{j} + \frac{df}{dr} \cdot \frac{z}{r}\hat{k}$

$= \frac{1}{r} \cdot \frac{df}{dr}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$

$\therefore \nabla f = f'(r) \cdot \frac{\bar{r}}{r}$ // Hence Proved. // ← Remember!

$\nabla f = 2r^4 \cdot \bar{r} = f'(r) \cdot \frac{\bar{r}}{r}$

$\therefore f'(r) = 2r^5$

$\therefore f(r) = \int 2r^5 = \frac{2r^6}{6} + c$

$f(r) = \frac{r^6}{3} + c$ //

→ $\boxed{\nabla r^n = n \cdot r^{n-2} \cdot \bar{r}}$; $\boxed{\nabla (e^{r^2}) = 2e^{r^2} \cdot \bar{r}}$

Q. $\nabla\left[\frac{\bar{a}\cdot\bar{r}}{r^n}\right] = \frac{\bar{a}}{r^n} - \frac{n\cdot(\bar{a}\cdot\bar{r})\bar{r}}{r^{n+2}}$ ← Prove.

Ans.- Let $\phi = \frac{\bar{a}\cdot\bar{r}}{r^n} = \frac{a_1x + a_2y + a_3z}{r^n}$

$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

but $\frac{\partial\phi}{\partial x} = \frac{r^n\cdot a_1 - (a_1x + a_2y + a_3z)\cdot n\cdot r^{n-1}\cdot \partial r/\partial x}{r^{2n}}$

$= \frac{r^n\cdot a_1 - (\bar{a}\cdot\bar{r})\cdot n\cdot r^{n-1}\cdot x/r}{r^{2n}}$

~~Similarly,~~ $\cancel{\frac{\partial\phi}{\partial y}} = \frac{a_1}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\cdot x}{r^{n+2}}$

Similarly, $\frac{\partial\phi}{\partial y} = \frac{a_2}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\cdot y}{r^{n+2}}$

$\frac{\partial\phi}{\partial z} = \frac{a_3}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\cdot z}{r^{n+2}}$

$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= \frac{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\cdot(x\hat{i} + y\hat{j} + z\hat{k})}{r^{n+2}}$

We know, $a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \bar{a}$, $x\hat{i} + y\hat{j} + z\hat{k} = \bar{r}$

$\therefore \nabla\phi = \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})\cdot\bar{r}}{r^{n+2}}$ //

Hence Proved. //

Q. Find directional derivative of $\phi = x^4 + y^4 + z^4$ at $A \equiv (1, -2, 1)$ in direction of AB, where $B \equiv (2, 6, -1)$

Ans. $D_{\hat{u}} \phi(A) = \nabla \phi(A) \cdot \hat{u}$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$= 4x^3 \hat{i} + 4x^3 \hat{j} + 4x^3 \hat{k}$$

$$\therefore \nabla \phi(A) = 4(1)^3 \hat{i} + 4(-2)^3 \hat{j} + 4(1)^3 \hat{k}$$

$$= 4\hat{i} - 32\hat{j} + 4\hat{k}$$

$$\bar{AB} = \bar{B} - \bar{A} = (1, 8, -2)$$

$$\therefore \hat{u} = \frac{\bar{AB}}{|\bar{AB}|} = \frac{(1, 8, -2)}{\sqrt{69}}$$

$$\therefore D_{\hat{u}} \phi(A) = (4\hat{i} - 32\hat{j} + 4\hat{k}) \cdot \frac{(\hat{i} + 8\hat{j} - 2\hat{k})}{\sqrt{69}}$$

$$= \frac{4 - 256 - 8}{\sqrt{69}}$$

$$\therefore D_{\hat{u}} \phi(A) = \frac{-260}{\sqrt{69}}$$

Q. Find directional derivative of $\phi = x^2+y^2+z^2$ in direction of line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at $(1,2,3)$

Ans. $D_{\hat{u}}\,\phi(A) = \nabla\phi(A)\cdot\hat{u}$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$ $\cancel{= 2x+2y+2z}$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ at $A \equiv (1,2,3)$

$\therefore \nabla\phi(A) = 2\hat{i}+4\hat{j}+6\hat{k}$

Given direction is $3\hat{i}+4\hat{j}+5\hat{k}$

$\therefore \hat{u} = \frac{3\hat{i}+4\hat{j}+5\hat{k}}{\sqrt{3^2+4^2+5^2}}$

$\therefore D_{\hat{u}}\,\phi(A) = (2\hat{i}+4\hat{j}+6\hat{k})\cdot\frac{(3\hat{i}+4\hat{j}+5\hat{k})}{5\sqrt{2}}$

$= \frac{6+16+30}{5\sqrt{2}} = \frac{52}{5\sqrt{2}}$

Q. Find directional derivative of $\phi = e^{2x} \cdot \cos(yz)$ at $(0,0,0)$ in direction of tangent to curve $x = a\sin(t)$, $y = a\cos(t)$, $z = at$ at $t = \pi/4$.

Ans. $D_{\hat{u}}\phi(A) = \nabla\phi(A) \cdot \hat{u}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 2e^{2x}\cos(yz)\hat{i} - ze^{2x}\sin(yz)\hat{j} - ye^{2x}\sin(yz)\hat{k}$$

At $A \equiv (0,0,0)$,

$$\nabla\phi(A) = 2\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\bar{r} = a\sin(t)\hat{i} + a\cos(t)\hat{j} + at\hat{k}$$

$\therefore$ Tangent to $\bar{r} = \frac{d\bar{r}}{dt} = a\cos(t)\hat{i} - a\sin(t)\hat{j} + a\hat{k}$

Tangent at $t = \pi/4 \rightarrow \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$$\therefore \hat{u} = \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}}{\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}}\right)^2 + a^2}} = \frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}$$

$$\therefore D_{\hat{u}}\phi(A) = (2\hat{i} + 0\hat{j} + 0\hat{k}) \cdot \left(\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{\sqrt{2}}\right)$$

$$= \left(2 \times \frac{1}{2}\right) = 1$$

Q. Find directional derivative of $\phi = xy^2 + yz^3$ at $(2,-1,1)$ in direction of normal to surface $x\log z - y^2 + 4 = 0$, at point $(-1,2,1)$

Ans. $D_{\hat{u}}\,\phi(A) = \nabla\phi(A)\cdot\hat{u}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \cancel{(y^2+yz^3)}\; y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$$

At point $A \equiv (2,-1,1)$,

$$\nabla\phi(A) = \hat{i} - 3\hat{j} - 3\hat{k}$$

$$\boxed{\begin{array}{l}\text{If } \phi(x,y,z) = c \\ \text{Normal to surface} = \nabla\phi\end{array}}$$ ← Remember!

Let $\varphi = x\log z - y^2 + 4$

$$\therefore \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

$$= \log(z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$$

$$\therefore \text{At } (-1,2,1) \Rightarrow 0\hat{i} - 4\hat{j} - \hat{k} = \nabla\varphi$$

$$\therefore \hat{u} = \frac{-4\hat{j} - \hat{k}}{\sqrt{4^2+1^2}} = \frac{-4\hat{j} - \hat{k}}{\sqrt{17}}$$

$$\therefore D_{\hat{u}}\,\phi(A) = (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{(-4\hat{j} - \hat{k})}{\sqrt{17}}$$

$$\cancel{= 12\hat{j} + 3k} = \frac{15}{\sqrt{17}}$$

Q. Find angle between surfaces $x\log(z) + 1 - y^2 = 0$ and $x^2y + z = 2$ at $(1,1,1)$.

Ans. Let $\phi(x,y,z) = x\log(z) + 1 - y^2$
and $\psi(x,y,z) = x^2y + z - 2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \log(z)\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$$

$\therefore$ at $(1,1,1) \rightarrow \nabla\phi = -2\hat{j} + \hat{k}$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$$

$$= 2xy\hat{i} + x^2\hat{j} + \hat{k}$$

$\therefore$ at $(1,1,1) \rightarrow 2\hat{i} + \hat{j} + \hat{k}$

Let θ be angle between $\nabla\phi$ and $\nabla\psi$

$\therefore \nabla\phi \cdot \nabla\psi = |\nabla\phi| \cdot |\nabla\psi| \cdot \cos\theta$

$\therefore$ ~~$\cos\theta = |\nabla\phi| \cdot |\nabla\psi|$~~

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi| \cdot |\nabla\psi|}$$

$$= \frac{(-2\hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{j} + \hat{k})}{\sqrt{5} \cdot \sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$$

Q. Find values of a, b, c if directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has max. magnitude 64 in direction parallel to z-axis.

Ans. $\phi = axy^2 + byz + cz^2x^3$

$\therefore \nabla\phi = (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$

$\therefore$ At $(1,2,-1) \rightarrow \nabla\phi = (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k}$

$$|\nabla\phi| = \sqrt{(4a+3c)^2 + (4a-b)^2 + (2b-2c)^2} = 64 \text{ (given)}$$

vector parallel to z-axis is given by : $0\hat{i} + 0\hat{j} + \hat{k}$ (given direction)

Since $\nabla\phi$ and $\hat{k}$ are parallel,

$$\therefore \frac{4a+3c}{0} = \frac{4a-b}{0} = \frac{2b-2c}{1} = t$$

$4a+3c = 0$, $4a-b=0$, $2b-2c = t$

from these and $|\nabla\phi|$, we get

$\sqrt{t^2} = 64$ $\therefore t = 64$ $\therefore 2b-2c = 64$

Solving :

$b - c = 32$ — (i)

$4a+3c = 0$ — (ii)

$4a - b = 0$ — (iii)

$\therefore a = 6, b = 24, c = -8$

[Ans: $a=10, b=8, c=-2$]

Q. Find a, b, c if normal to the surface ~~a^2~~ $ax^2+bxz+z^2y=c$ at ~~a~~ $P(-1,1,2)$ is parallel to $x^2-y^2+2z=2$ at $(1,1,1)$.

Ans.

- Divergence and Curl :-

① Let $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

then $\text{div } \bar{f} = \nabla \cdot \bar{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$

If $\nabla \cdot \bar{f} = 0$ then $\bar{f}$ is called solenoidal.

② If $\bar{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

then $\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

If $\nabla \times \bar{f} = \bar{0}$, then $\bar{f}$ is called irrotational.

→ Identities:

1) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

2) $\nabla \cdot (\phi\bar{f}) = \phi(\nabla \cdot \bar{f}) + \bar{f} \cdot \nabla\phi$

3) $\nabla \cdot (\bar{f} \times \bar{g}) = \bar{g} \cdot (\nabla \times \bar{f}) - \bar{f} \cdot (\nabla \times \bar{g})$

4) $\nabla \times (\phi\bar{f}) = \phi(\nabla \times \bar{f}) + \nabla\phi \times \bar{f}$

Q. If $\bar{a}$ is a constant vector $|\bar{a}| = a$, prove that

$$\nabla \cdot \{(\bar{a} \cdot \bar{r})\bar{a}\} = a^2$$

$$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\bar{a} \cdot \bar{r} = a_1x + a_2y + a_3z = \phi$$

$$\therefore \nabla \cdot (\phi\bar{a}) = \phi(\nabla \cdot \bar{a}) + \bar{a} \cdot (\nabla\phi)$$

$$= (a_1x + a_2y + a_3z)0 + \bar{a} \cdot \bar{a}$$

$$= |\bar{a}|^2 = a^2$$

$$\nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} = 0$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \bar{a}$$

Q. $\nabla\left\{\nabla\cdot\frac{\bar{r}}{r}\right\} = \frac{-2\bar{r}}{r^3}$

Ans. Let $\phi = \frac{1}{r}$, $\bar{f} = \bar{r}$

but $\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$

$\therefore \nabla\cdot\left(\frac{\bar{r}}{r}\right) = \frac{1}{r}(3) + \bar{r}\cdot\left(\frac{-1}{r^2}\right)\frac{\bar{r}}{r}$ $\quad\left[\nabla f(r) = f'(r)\cdot\frac{\bar{r}}{r}\right]$

$= \frac{3}{r} - \frac{\bar{r}\cdot\bar{r}}{r^3} = \frac{3}{r} - \frac{r^2}{r^3}$

$\therefore \nabla\cdot\left(\frac{\bar{r}}{r}\right) = \frac{2}{r}$

$\therefore \nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right] = \nabla\left(2/r\right)$

$= \frac{-2}{r^2}\cdot\frac{\bar{r}}{r}$

$\therefore \nabla\left[\nabla\cdot\frac{\bar{r}}{r}\right] = \frac{-2\bar{r}}{r^3}$ //

Q. Prove that $\nabla\cdot\left(r\nabla\frac{1}{r^n}\right) = \frac{n(n-2)}{r^{n+1}}$

Ans. $\nabla\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\cdot\frac{\bar{r}}{r}$ $\left(\text{from } \nabla f(r) = f'(r)\cdot\frac{\bar{r}}{r}\right)$

$\therefore\ r\nabla\left(\frac{1}{r^n}\right) = \frac{-n\bar{r}}{r^{n+1}}$

Now, we know,

$$\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot\nabla\phi$$

$$\therefore\ \nabla\cdot\left(r\nabla\left(\frac{1}{r^n}\right)\right) = \nabla\cdot\left(\frac{-n\bar{r}}{r^{n+1}}\right)$$

$$= -n\left[\frac{1}{r^{n+1}}\underbrace{\nabla\cdot\bar{r}}_{(=3)} + \bar{r}\cdot\nabla\left(\frac{1}{r^{n+1}}\right)\right]$$

$(=3) \rightarrow$ we've done before

$$= -n\left[\frac{3}{r^{n+1}} + \bar{r}\cdot\frac{-(n+1)}{r^{n+2}}\frac{\bar{r}}{r}\right] \qquad (\bar{r}\cdot\bar{r} = r^2)$$

$$= -n\left[\frac{3}{r^{n+1}} - \frac{(n+1)}{r^{n+1}}\right] = -n\frac{(2-n)}{r^{n+1}}$$

$$\therefore\ \nabla\cdot\left(r\nabla\left(\frac{1}{r^n}\right)\right) = \frac{n(n-2)}{r^{n+1}}$$

//

Q. $\nabla\cdot\left\{\frac{f(r)}{r}\bar{r}\right\} = \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$ ← Prove

Hence or otherwise Prove that $\nabla\cdot(r^n\bar{r}) = (n+3)r^n$.

Ans. Let $\phi = \frac{f(r)}{r}$, and $\bar{f} = \bar{r}$

but $\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot\nabla\phi$

$= \frac{f(r)}{r}\underbrace{\nabla\cdot\bar{r}}_{=3} + \bar{r}\cdot\nabla\frac{f(r)}{r}$ ← ~~difference~~ ($\frac{u}{v}$ rule)

$\nabla\cdot(\phi\bar{f}) = \frac{3f(r)}{r} + \bar{r}\cdot\left[\frac{rf'(r)-f(r)}{r^2}\right]\frac{\bar{r}}{r}$ $(\bar{r}\cdot\bar{r} = r^2)$

since $\left[\nabla f(r) = f'(r)\frac{\bar{r}}{r}\right]$

$\therefore \nabla\cdot\left(\frac{f(r)}{r}\bar{r}\right) = \frac{3f(r)}{r} + f'(r) - \frac{f(r)}{r}$

$= \frac{2f(r)}{r} + f'(r) = \text{LHS}$

Taking RHS, $\frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right] = \frac{1}{r^2}\left[r^2 f'(r) + 2rf(r)\right]$

$\therefore \text{RHS} = \frac{2f(r)}{r} + f'(r) = \text{LHS}$

LHS = RHS Hence Proved.

$\nabla\cdot(r^n\bar{r}) \rightarrow$ Let $\frac{f(r)}{r} = r^n$ $\therefore f(r) = r^{n+1}$

$\therefore \nabla\cdot(r^n\bar{r}) = \frac{1}{r^2}\frac{d}{dr}(r^2 r^{n+1}) = \frac{1}{r^2}\frac{d}{dr}(r^{n+3})$

$\therefore \nabla\cdot(r^n\bar{r}) = \frac{(n+3)r^{n+2}}{r^2} = (n+3)r^n$

Hence Proved.

Q. Find $f(r)$ so that $f(r)\bar{r}$ is both solenoidal and irrotational.

Ans. For solenoidal, $\nabla\cdot(f(r)\bar{r}) = 0$

$\therefore \nabla\cdot(f(r)\bar{r}) = f(r)\,\nabla\cdot\bar{r} + \bar{r}\cdot\nabla f(r)$ (same property as previous questions)

$\therefore \nabla\cdot(f(r)\bar{r}) = 3f(r) + \bar{r}\cdot f'(r)\dfrac{\bar{r}}{r}$

$= 3f(r) + \dfrac{r^2}{r}f'(r) = 0$

$\therefore 3f(r) = -rf'(r)$

$-\dfrac{3}{r} = \dfrac{f'(r)}{f(r)}$ $\qquad \therefore -\displaystyle\int \frac{3}{r}\,dr = \int \frac{f'(r)}{f(r)}\,dr$

$\therefore -3\log(r) = \log(f(r)) + \log(c_1)$

~~$\log(r^{-3}) - \log(f(r)) = \log(c_1)$~~

$\log(f(r)) + 3\log(r) = \log(c_1)$

$\therefore \log(f(r)\cdot r^3) = \log(c)$

$\therefore r^3 f(r) = c$ $\qquad f(r) = \dfrac{c}{r^3}$

(Do irrotational)

(sir told okay if skipped in exam)

Q. Prove that $\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is solenoidal, and determine contants a, b, c if $\bar{F}$ is irrotational.

Ans. Solenoidal $\rightarrow \nabla \cdot \bar{F} = 0$

$$\therefore \nabla \cdot \bar{F} = \frac{\partial(x+2y+az)}{\partial x} + \frac{\partial(bx-3y-z)}{\partial y} + \frac{\partial(4x+cy+2z)}{\partial z}$$

$$= 1 - 3 + 2 = 0$$

Hence Proved $\bar{F}$ is solenoidal.

If $\bar{F}$ is irrotational $\rightarrow \nabla \times \bar{F} = \bar{0}$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$$\therefore \hat{i}(c-(-1)) - \hat{j}(4-a) + \hat{k}(b-2) = \bar{0}$$

$$\therefore (c+1)\hat{i} + (a-4)\hat{j} + (2-b)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$c+1=0 \rightarrow \therefore c=-1$

$a-4=0 \rightarrow \therefore a=4$

$2-b=0 \rightarrow \therefore b=2$

$\therefore a=4, b=2, c=-1$

Q. If $\overline{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$, find scalar potential ϕ such that $\overline{F} = \nabla\phi$ and $\phi(1,0,1) = 8$.

Ans. ϕ is called scalar potential of $\overline{F}$, if $\overline{F} = \nabla\phi$

But $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\therefore \frac{\partial\phi}{\partial x} = y^2 - 2xyz^3$; $\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3$; $\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2$

We know $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\therefore d\phi = (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz$$

$$= (y^2dx + 2xy\,dy) - (2xyz^3dx + x^2z^3dy + 3x^2yz^2dz) + 3dy + 6z^3dz$$

$$\therefore d\phi = d(xy^2) - d(x^2yz^3) + d(3y) + d\left(\frac{6z^4}{4}\right)$$

$$\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3z^4}{2} + c$$

But $\phi(1,0,1) = 8$

$\therefore 8 = \frac{3}{2} + c \quad \therefore c = \frac{13}{2}$

$$\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3z^4}{2} + \frac{13}{2}$$