Vector Algebra

• **Scalar Triple Product** ↓

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be any 3 vectors. Then, $(\bar{a}\times\bar{b})\cdot\bar{c}$ is called scalar triple product of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

$$\boxed{(\bar{a}\times\bar{b})\cdot\bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\bar{a}\ \bar{b}\ \bar{c}]}$$

→ **Properties:**

① $(\bar{a}\times\bar{b})\cdot\bar{c} = (\bar{b}\times\bar{c})\cdot\bar{a} = (\bar{c}\times\bar{a})\cdot\bar{b}$ (cyclic)

② $(\bar{a}\times\bar{b})\cdot\bar{c} = \bar{a}\cdot(\bar{b}\times\bar{c})$

③ $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{a}\ \bar{c}\ \bar{b}] = -[\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}]$

④ $[\bar{a}\ \bar{a}\ \bar{b}] = [\bar{a}\ \bar{b}\ \bar{a}] = [\bar{b}\ \bar{a}\ \bar{a}] = 0$

⑤ If $\bar{a}, \bar{b}, \bar{c}$ are coplanar, then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$.

⑥ Volume of tetrahedron (if $\bar{a}, \bar{b}, \bar{c}$ are sides with same initial point),
$= \frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$

⑦ Volume of parallelopiped, whose coterminal edges are $\bar{a}, \bar{b}, \bar{c}$,
$= [\bar{a}\ \bar{b}\ \bar{c}]$

Q. Show that $(\bar{p}+\bar{q})\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})] = 2[\bar{p}\,\bar{q}\,\bar{r}]$

Ans. LHS $= \bar{p}\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})] + \bar{q}\cdot[(\bar{q}+\bar{r})\times(\bar{r}+\bar{p})]$

$= \bar{p}\cdot[\bar{q}\times(\bar{r}+\bar{p})+\bar{r}\times(\bar{r}+\bar{p})] + \bar{q}[\bar{q}\times(\bar{r}+\bar{p})+\bar{r}\times(\bar{r}+\bar{p})]$.

$= \bar{p}\cdot[(\bar{q}\times\bar{r})+(\bar{q}\times\bar{p})+(\bar{r}\times\bar{r})+(\bar{r}\times\bar{p})]$

$+\bar{q}[(\bar{q}\times\bar{r})+(\bar{q}\times\bar{p})+(\bar{r}\times\bar{r})+(\bar{r}\times\bar{p})]$ $\quad \boxed{\bar{r}\times\bar{r}=0}$

$= [\bar{p}\,\bar{q}\,\bar{r}] + [\bar{p}\,\bar{q}\,\bar{p}] + [\bar{p}\,\bar{r}\,\bar{p}]$

$+ [\bar{q}\,\bar{q}\,\bar{r}] + [\bar{q}\,\bar{q}\,\bar{p}] + [\bar{q}\,\bar{r}\,\bar{p}]$

But $[\bar{a}\,\bar{a}\,\bar{b}] = 0$

$\therefore$ LHS $= [\bar{p}\,\bar{q}\,\bar{r}] + [\bar{q}\,\bar{r}\,\bar{p}]$

But $[\bar{q}\,\bar{r}\,\bar{p}] = [\bar{p}\,\bar{q}\,\bar{r}]$ (cyclic)

$\therefore$ LHS $= 2[\bar{p}\,\bar{q}\,\bar{r}]$ = RHS

Hence Proved.

- Vector Triple Product

Let $\bar{a}, \bar{b}, \bar{c}$ be any 3 vectors. Then vector triple product of $\bar{a}, \bar{b}, \bar{c}$ is $(\bar{a}\times\bar{b})\times\bar{c}$

$$\boxed{(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}}$$

Q. If $\bar{a}, \bar{b}, \bar{c}$ are coplanar vectors, then prove that $(\bar{a}\times\bar{b}), (\bar{b}\times\bar{c}), (\bar{c}\times\bar{a})$ are also coplanar vectors.

Ans. Given $[\bar{a}\ \bar{b}\ \bar{c}] = 0$,

we need to prove $[\bar{a}\times\bar{b}\quad \bar{b}\times\bar{c}\quad \bar{c}\times\bar{a}] = \bar{0}$

$$\therefore (\bar{a}\times\bar{b})\cdot[(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})] = 0$$

Let $\bar{c}\times\bar{a} = \bar{m}$.

$$\therefore (\bar{a}\times\bar{b})\cdot[(\bar{b}\times\bar{c})\times\bar{m}]$$

$$= (\bar{a}\times\bar{b})\cdot[(\bar{b}\cdot m)\bar{c} - (\bar{c}\cdot\bar{m})\bar{b}]$$

$$= (\bar{a}\times\bar{b})\cdot[(\bar{b}\cdot(\bar{c}\times\bar{a}))\bar{c} - \underbrace{(\bar{c}\cdot(\bar{c}\times\bar{a}))}_{=0}\bar{b}]$$

$$= (\bar{a}\times\bar{b})\cdot\underbrace{[\bar{a}\ \bar{b}\ \bar{c}]}_{\text{Scalar}}\cdot\bar{c}$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]\left((\bar{a}\times\bar{b})\cdot\bar{c}\right) = [\bar{a}\ \bar{b}\ \bar{c}]\cdot[\bar{a}\ \bar{b}\ \bar{c}]$$

$$= [\bar{a}\ \bar{b}\ \bar{c}]^2$$

But $[\bar{a}\ \bar{b}\ \bar{c}] = 0$,

$$\therefore [\bar{a}\times\bar{b}\quad \bar{b}\times\bar{c}\quad \bar{c}\times\bar{a}] = 0$$

Q. Show that $[\bar{d} \times (\bar{a} \times \bar{b})] \cdot (\bar{a} \times \bar{c}) = [\bar{a}\,\bar{b}\,\bar{c}](\bar{a} \cdot \bar{d})$

Ans. $LHS = [(\bar{d} \cdot \bar{b})\bar{a} - (\bar{d} \cdot \bar{a})\bar{b}] \cdot (\bar{a} \times \bar{c})$

$= [\underbrace{(\bar{d} \cdot \bar{b})}_{\text{scalar}}\bar{a}] \cdot (\bar{a} \times \bar{c}) - [\underbrace{(\bar{d} \cdot \bar{a})}_{\text{scalar}}\bar{b}] \cdot (\bar{a} \times \bar{c})$

$= [\bar{d} \cdot \bar{b}]\underbrace{(\bar{a} \cdot (\bar{a} \times \bar{c}))}_{=0} - [\bar{d} \cdot \bar{a}](\bar{b} \cdot (\bar{a} \times \bar{c}))$

$LHS = -[\bar{b}\,\bar{a}\,\bar{c}](\bar{a} \cdot \bar{d})$

$= -[\bar{a}\,\bar{c}\,\bar{b}](\bar{a} \cdot \bar{d})$

$= [\bar{a}\,\bar{b}\,\bar{c}] \cdot (\bar{a} \cdot \bar{d})$ // $= RHS$

Hence Proved //

Q. $(\bar{a}\times\bar{b})\cdot(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a}) = [\bar{a}\cdot(\bar{b}\times\bar{c})]^2$

Ans.

Q. Vectors $\bar{u}, \bar{v}, \bar{w}$ are non-coplanar, then show that $(\bar{u}\times\bar{v}), (\bar{v}\times\bar{w}), (\bar{w}\times\bar{u})$ are also non-coplanar. Hence obtain scalars l, m, n such that ~~$\bar{u} = l(\bar{v}$~~
$\bar{u} = l(\bar{v}\times\bar{w}) + m(\bar{w}\times\bar{u}) + n(\bar{u}\times\bar{v})$

Ans. [First ~~part~~ part same as question on page 93. !]

$\therefore$ ~~$(\bar{u}\times\bar{v})$~~ $[\bar{u}\times\bar{v} \quad \bar{v}\times\bar{w} \quad \bar{w}\times\bar{u}] = [\bar{u}\,\bar{v}\,\bar{w}]^2 \neq 0$ //

Taking 2nd part:

$\bar{u} = l(\bar{v}\times\bar{w}) + m(\bar{w}\times\bar{u}) + n(\bar{u}\times\bar{v})$

$\therefore \bar{u}\cdot\bar{u} = l(\bar{v}\times\bar{w})\cdot\bar{u} + m\underbrace{(\bar{w}\times\bar{u})\cdot\bar{u}}_{=0} + n\underbrace{(\bar{u}\times\bar{v})\cdot\bar{u}}_{=0}$

$\therefore l = \dfrac{\bar{u}\cdot\bar{u}}{[\bar{u}\,\bar{v}\,\bar{w}]}$ //

Similarly, $m = \dfrac{\bar{u}\cdot\bar{v}}{[\bar{u}\,\bar{v}\,\bar{w}]}$ //, $n = \dfrac{\bar{u}\cdot\bar{w}}{[\bar{u}\,\bar{v}\,\bar{w}]}$ //

Scalar Product of 4 vectors

$(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$ is called Scalar product of 4 vectors, and is given by:

$$\boxed{(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix}}$$ ← Lagrange's Identity

Q. Prove that $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) + (\bar{b}\times\bar{c})\cdot(\bar{a}\times\bar{d}) + (\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{d}) = 0$

Ans. $\because$ LHS $= \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{b}\cdot\bar{a} & \bar{c}\cdot\bar{a} \\ \bar{b}\cdot\bar{d} & \bar{c}\cdot\bar{d} \end{vmatrix} + \begin{vmatrix} \bar{c}\cdot\bar{b} & \bar{a}\cdot\bar{b} \\ \bar{c}\cdot\bar{d} & \bar{a}\cdot\bar{d} \end{vmatrix}$

$$= \left[(\bar{a}\cdot\bar{c})(\bar{b}\cdot\bar{d}) - (\bar{a}\cdot\bar{d})(\bar{b}\cdot\bar{c})\right] + \left[(\bar{b}\cdot\bar{a})(\bar{c}\cdot\bar{d}) - (\bar{b}\cdot\bar{d})(\bar{c}\cdot\bar{a})\right] + \left[(\bar{c}\cdot\bar{b})(\bar{a}\cdot\bar{d}) - (\bar{c}\cdot\bar{d})(\bar{a}\cdot\bar{b})\right]$$

$= 0$ (all terms get cancelled) //

- Vector Product of 4 Vectors :

$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$ is vector product of 4 vectors.

$$\boxed{\begin{aligned}(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) &= [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a} \\ &= [\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d}\end{aligned}}$$

Q. Prove that $[(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{d} = (\bar{a} \cdot \bar{d})[\bar{a}\,\bar{b}\,\bar{c}]$

Ans. $LHS = \left(\underbrace{[\bar{a}\,\bar{a}\,\bar{c}]}_{=0}\bar{b} - [\bar{b}\,\bar{a}\,\bar{c}]\bar{a}\right) \cdot \bar{d}$

$= [\bar{a}\,\bar{b}\,\bar{c}](\bar{a} \cdot \bar{d}) = RHS$

Hence Proved.

Q. $\bar{d}\cdot[\bar{a}\times[\bar{b}\times(\bar{c}\times\bar{d})]] = (\bar{b}\cdot\bar{d})\,[\bar{a}\;\bar{c}\;\bar{d}]$

Ans. $LHS = \bar{d}\cdot\left[\bar{a}\times\left[(\bar{b}\cdot\bar{d})\bar{c}-(\bar{b}\cdot\bar{c})\bar{d}\right]\right]$

$= \bar{d}\cdot\left[(\bar{b}\cdot\bar{d})(\bar{a}\times\bar{c})-(\bar{b}\cdot\bar{c})(\bar{a}\times\bar{d})\right]$ (since $\bar{b}\cdot\bar{d}$ and $\bar{b}\cdot\bar{c}$ are scalar quantities)

$= (\bar{b}\cdot\bar{d})\left[\bar{d}\cdot(\bar{a}\times\bar{c})\right]-(\bar{b}\cdot\bar{c})\left[\bar{d}\cdot(\bar{a}\times\bar{d})\right]$

$= (\bar{b}\cdot\bar{d})\left[\bar{d}\;\bar{a}\;\bar{c}\right]-(\bar{b}\cdot\bar{c})\underbrace{\left[\bar{d}\;\bar{a}\;\bar{d}\right]}_{=0}$

$= (\bar{b}\cdot\bar{d})\left[\bar{a}\;\bar{c}\;\bar{d}\right]$ $= RHS$

Hence Proved.