Laplace Transform

(is an operator)*

* Let $f(t)$ be a function of $t \geq 0$, then Laplace of $f(t)$.

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

[t = time (time domain)
s = frequency (freq. domain)]

$$= \phi(s) = \bar{f}(s)$$

Q. Find Laplace transform of $f(t) = \begin{cases} \cos(t) & \text{for } 0<t<\pi \\ \sin(t) & \text{for } t>\pi \end{cases}$

Ans. $$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos(t)\, dt + \int_\pi^\infty e^{-st} \sin(t)\, dt$$

☆☆ [we know,

$$\int e^{ax} \sin(bx)\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right)$$

$$\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right)$$]

$$\therefore L[f(t)] = \left[\frac{e^{-st}}{s^2+1}\left(-s\cos(t)+\sin(t)\right)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}\left(-s\sin(t) - \cos(t)\right)\right]_\pi^\infty$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(-s(-1)) - \frac{1}{s^2+1}(-s)\right] + \left[0 - \frac{e^{-\pi s}}{s^2+1}(0-(-1))\right]$$

$$= \frac{s \cdot e^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1} \quad //$$

[Even if not defined at interval endpoints, integration works (Riemann).]

- Laplace for standard functions ↓

→ $L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at}\, dt$

$= \int_0^\infty e^{-(s-a)t}\, dt$

$= -\left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty$

$= \frac{1}{s-a}$ (for $s > a$)

Parallely,

→ $L[e^{-at}] = \frac{1}{s+a}$ (for $s > -a$)

→ Linearity Property:

$$\boxed{L[a \cdot f(t) + b \cdot g(t)] = a \cdot L[f(t)] + b \cdot L[g(t)]}$$

→ $L[\cos(at)] = ?$

$L[\sin(at)] = ?$

∴ We have $e^{iat} = \cos(at) + i\sin(at)$

$\therefore L[e^{iat}] = L[\cos(at) + i\sin(at)]$

$\therefore \frac{1}{s-ia} = L[\cos(at)] + i\,L[\sin(at)]$

$\therefore \frac{s+ia}{s^2+a^2} = L[\cos(at)] + i \cdot L[\sin(at)]$ (Comparing real and imaginary parts)

$\therefore \boxed{L[\cos(at)] = \frac{s}{s^2+a^2}}$, $\boxed{L[\sin(at)] = \frac{a}{s^2+a^2}}$

→ $L[\cosh(at)] = ?$

$L[\sinh(at)] = ?$

$$\therefore L[\cosh(at)] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left(L[e^{at}] + L[e^{-at}]\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{s}{s^2-a^2}$$

~~Parallely,~~ Similarly,

$$L[\sinh(at)] = \frac{a}{s^2-a^2}$$

→ $L[t^n] = \begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}} \\ \dfrac{n!}{s^{n+1}}, \quad n = 0, 1, 2 \ldots \end{cases}$

$$L[t^n] = \int_0^\infty e^{-st} t^n \cdot dt$$

Let $st = u$, $\therefore t = u/s$, $dt = du/s$

$$\therefore L[t^n] = \int_0^\infty e^{-u} \cdot \frac{u^n}{s^n} \cdot \frac{du}{s} = \frac{1}{s^{n+1}} \cdot \underbrace{\int_0^\infty e^{-u} u^n du}_{(\text{Gamma})}$$

$$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

→ $L[1] = \frac{1}{s}$

→ $L[K] = \frac{K}{s}$

Q. Find $L[t^2 - e^{-2t} + \cosh^2(3t) + \sin(3t)]$

Ans. $= L[t^2] - L[e^{-2t}] + L[\cosh^2(3t)] + L[\sin(3t)]$

$= \frac{\lfloor 3}{s^3} - \frac{1}{s+2} + \frac{3}{s^2+9} + L\left[\frac{1+\cosh(6t)}{2}\right]$

$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{3}{s^2+9} + \frac{1}{2}\left[L(1) + L(\cosh(6t))\right]$

$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{3}{s^2+9} + \frac{1}{2s} + \frac{s}{2s^2-72}$

Q. Find $L[\sin^5(t)]$

Ans. Let $z = \cos(t) + i\sin(t)$ $\quad \therefore \frac{1}{z} = \cos(t) - i\sin(t)$

$\left(z - \frac{1}{z}\right) = 2i\sin(t)$ $\quad \therefore \left(z - \frac{1}{z}\right)^5 = 2^5 . i . \sin^5(t)$

$\therefore z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5} = 2^5 . i . \sin^5(t)$

$\therefore = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$

$\therefore 2\cancel{i}\sin(5t) - 10\cancel{i}\sin(3t) + 20\cancel{i}\sin(t) = 2^5 . \cancel{i} . \sin^5(t)$

$\therefore \sin^5(t) = \frac{1}{2^4}\left[\sin(5t) - 5\sin(3t) + 10\sin(t)\right]$

$\therefore L[\sin^5(t)] = \frac{1}{2^4}\left[\frac{5}{s^2+25} - 5 \cdot \frac{3}{s^2+9} + 10 \cdot \frac{1}{s^2+1}\right]$

$L[\sin^5(t)] = \frac{1}{2^4}\left[\frac{5}{s^2+25} - \frac{15}{s^2+9} + \frac{10}{s^2+1}\right] = \frac{120}{(s^2+1)(s^2+9)(s^2+25)}$

find out!

Q. $x = \underbrace{\cos(t)\cdot\cos(2t)}\cdot\cos(3t)$. Find $L[x]$

Ans.

$$= \frac{1}{2}\left[\cos(3t)+\cos(t)\right]\cdot\cos(3t)$$

$$= \frac{1}{2}\left[\cos(3t)\cdot\cos(3t) + \cos(t)\cdot\cos(3t)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1+\cos(6t)}{2}\right) + \frac{1}{2}\left(\cos(4t)+\cos(2t)\right)\right]$$

$$= \frac{1}{4}\left[1+\cos(6t)+\cos(4t)+\cos(2t)\right]$$

$$\therefore L[\cos(t)\cdot\cos(2t)\cdot\cos(3t)] = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

Q. Find $L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right]$.

Ans. $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$

$$\therefore \frac{\cos(\sqrt{t})}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots$$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\overline{|1/2}}{s^{1/2}} - \frac{1}{2!}\frac{\overline{|3/2}}{s^{3/2}} + \frac{1}{4!}\frac{\overline{|5/2}}{s^{5/2}} - \frac{1}{6!}\frac{\overline{|7/2}}{s^{7/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \left(\frac{1}{4s}\right) + \frac{3}{96}\cdot\frac{1}{s^2} - \frac{15}{8\times6\times5!}\cdot\frac{1}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \frac{1}{4!}\left(\frac{1}{4s}\right)^4 - \cdots\right]$$

Let $1/4s = x$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

$$L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}\cdot e^{-x}$$

$$\therefore L\left[\frac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}}\cdot e^{-1/4s}$$

- Change of scale property :

$$\text{If } L[f(t)] = \phi(s),$$
$$\text{then } L[f(at)] = \frac{1}{a}\phi(s/a)$$

Example: If $L[\text{erf}(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$,

Find $L[\text{erf}(2\sqrt{t})]$.

Solution: $L[\text{erf}(2\sqrt{t})] = L[\text{erf}(\sqrt{4t})]$

Let $f(t) = \text{erf}(\sqrt{t})$

$\therefore f(4t) = \text{erf}(\sqrt{4t})$

$$\therefore L[\text{erf}(\sqrt{4t})] = \frac{1}{4}\cdot\phi(s/4)$$

$$= \frac{1}{\cancel{4}}\cdot\frac{1}{s/\cancel{4}\sqrt{s/4+1}}$$

$$\therefore L[\text{erf}(\sqrt{4t})] = \frac{2}{s\sqrt{s+4}}$$

- First Shifting Property ↧

If $L[f(t)] = \phi(s)$,
then $L[e^{at}\cdot f(t)] = \phi(s-a)$
also, $L[e^{-at}\cdot f(t)] = \phi(s+a)$

Example: Prove that $L[\sinh(t/2)\cdot \sin(\sqrt{3}t/2)] = \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+s^2+1}\right]$

Solution:

$$LHS = \sinh(t/2)\cdot \sin(\sqrt{3}t/2)$$

$$= \left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\cdot \sin(\sqrt{3}t/2)$$

$$= \frac{1}{2}\left[e^{t/2}\cdot \sin(\sqrt{3}t/2) - e^{-t/2}\cdot \sin(\sqrt{3}t/2)\right]$$

Let $\sin(\sqrt{3}t/2) = f(t)$

$$\therefore L[\sinh(t/2)\cdot \sin(\sqrt{3}t/2)] = \frac{1}{2}\left(L[e^{t/2}\cdot f(t)] - L[e^{-t/2}\cdot f(t)]\right)$$

$$= \frac{1}{2}\left(\phi(s-1/2) - \phi(s+1/2)\right)$$

$$= \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s-1/2)^2+3/4} - \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{2s}{(s^2+1)^2-s^2}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+s^2+1}\right]$$ //

Hence Proved.//

- **Second Shifting Property** ↓

If $L[f(t)] = \phi(s)$

and $g(t) = \begin{cases} \cancel{f(\ )} f(t-a) & (\text{for } t > a) \\ 0 & (\text{for } t < a) \end{cases}$ $[f(t-a)]$

then $L[g(t)] = e^{-as}\phi(s)$

- **Effect of Multiplication by t** ↓

If $L[f(t)] = \phi(s)$,

then $L[t^n \cdot f(t)] = (-1)^n \cdot \dfrac{d^n}{ds^n}[\phi(s)]$

If;

$n=1$: $L[t^1 \cdot f(t)] = -\dfrac{d}{ds}\phi(s) = -\phi'(s)$

$n=2$: $L[t^2 \cdot f(t)] = \dfrac{d^2}{ds^2}\phi(s) = \phi''(s)$

Q. Find $L[t \cdot e^{3t} \cdot \sin(4t)]$

Ans. $L[e^{3t} \cdot (t \cdot \sin(4t))]$

$= L[e^{3t} \cdot f(t)] \quad = \phi(s-3)$

$\therefore \phi(s) = L[f(t)]$

$= L[t \cdot \sin(4t)]$

$= -\dfrac{d}{ds}\left(L[\sin(4t)]\right)$

$= -\dfrac{d}{ds}\left(\dfrac{4}{s^2+16}\right)$

$= \dfrac{8s}{(s^2+16)^2}$

$\therefore \phi(s-3) = \dfrac{8(s-3)}{((s-3)^2+16)^2}$ //

Q. Find $L[e^{3t} \cdot t \cdot \sqrt{1+\sin t}]$

Ans. $\therefore L[e^{3t} \cdot f(t)] = \phi(s-3)$

$\therefore \phi(s) = L[f(t)] \quad = L[t \cdot \sqrt{1+\sin t}]$

$= -\dfrac{d}{ds}\left(L[\sqrt{1+\sin t}]\right)$

$= -\dfrac{d}{ds}\left(L[\sin(t/2) + \cos(t/2)]\right)$

$= -\dfrac{d}{ds}\left(\dfrac{1/2}{s^2+1/4} + \dfrac{s}{s^2+1/4}\right)$ (complete!!)

$= \dfrac{1}{2}\left(\dfrac{2s}{(s^2+1/4)^2}\right) + \dfrac{1}{s^2+1/4}$

Q. Find $L[t^3.\cos t]$ at $s=3$

Ans. $\therefore L[t^3.\cos t] = \frac{-d^3}{ds^3}\left(L[\cos t]\right)$

$$= \frac{-d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

$$= -\left[\frac{(s^2+1)^3.(6s^2-6)-(2s^3-6s).3.(s^2+1).2s}{(s^2+1)^6}\right]$$

$= \phi(s)$

$\therefore$ At $s=3$,

$$\phi(3) = -\left[\frac{10^3.48-(36.3.10.2.3)}{10^6}\right]$$

$$\phi(3) = \frac{21}{1250}$$

- Effect of Division by t

If $L[f(t)] = \phi(s)$,

then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s).ds$

also, $L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty\int_s^\infty(\phi(s).ds).ds$

Q. Find $L\left[\dfrac{e^{-2t}.\sin(2t)\cosh(t)}{t}\right]$

Ans. Let $f(t) = e^{-2t}.\sin(2t).\cosh(t)$

$$= e^{-2t}.\left(\frac{e^{t}+e^{-t}}{2}\right).\sin(2t)$$

$$= \frac{1}{2}\left(e^{-t}+e^{-3t}\right).\sin(2t)$$

$$\therefore L[f(t)] = \frac{1}{2}\left(L[e^{-t}.\sin(2t)] + L[e^{-3t}.\sin(2t)]\right)$$

$$\cancel{= \frac{1}{2}\left(\phi(s+1) + \phi(s+3)\right)}$$

[Using first-shifting property and $L[\sin(2t)]$]

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{(s+1)^2+4} + \frac{\cancel{2}}{(s+3)^2+4}\right)$$

$$= \phi(s)$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^{\infty}\phi(s).ds$$

$$= \int_s^{\infty}\left[\frac{1}{(s+1)^2+4}.ds + \frac{1}{(s+3)^2+4}.ds\right]$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \left[\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\pi}{2}\right) - \left(\tan^{-1}\left(\frac{s+1}{2}\right) + \tan^{-1}\left(\frac{s+3}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\cot^{-1}\left(\frac{s+1}{2}\right) + \cot^{-1}\left(\frac{s+3}{2}\right)\right] //$$

Q. Find $L\left[\frac{\sin^2 t}{t}\right]$

Hence prove that $\int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} \cdot dt = \frac{1}{4}\log_e(5)$

Ans. Let $f(t) = \sin^2(t)$

$$= \frac{1}{2}(1 - \cos(2t))$$

$$\therefore L[f(t)] = \frac{1}{2}\left(L[1] - L[\cos(2t)]\right)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \phi(s)$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s) \cdot ds$$

$$= \frac{1}{2}\left[\log(s) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

(since cannot directly put ∞)

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left(\log\left(\frac{s^2}{s^2+4}\right)\right)_s^\infty = \frac{1}{4}\left(\log\left(\frac{1}{1+4/s^2}\right)\right)_s^\infty$$

$$\therefore L\left[\frac{f(t)}{t}\right] = \frac{1}{4}\left(0 - \log\left(\frac{1}{1+4/s^2}\right)\right) = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right) = L\left[\frac{\sin^2 t}{t}\right]$$

By first definition, $L\left[\frac{\sin^2 t}{t}\right] = \int_0^\infty e^{-st} \cdot \frac{\sin^2(t)}{t} \cdot dt$

Putting $s = 1$

$\int_0^\infty e^{-t} \cdot \frac{\sin^2(t)}{t} \cdot dt$ (original question)

$\therefore$ Putting $s=1$, $\frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log_e(5)$ // Hence Proved. //

Q. Evaluate $\int_0^\infty \frac{\cos(6t) - \cos(4t)}{t} \cdot dt$

Ans Finding $L\left[\frac{\cos(6t) - \cos(4t)}{t}\right]$

Let $f(t) = \cos(6t) - \cos(4t)$

$\therefore L[f(t)] = L[\cos(6t)] - L[\cos(4t)]$

$= \frac{s}{s^2+36} - \frac{s}{s^2+16} = \phi(s)$

$\therefore L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s) \cdot ds$

$= \int_s^\infty \frac{s}{s^2+36} \cdot ds - \int_0^\infty \frac{s}{s^2+16} \cdot ds$

$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(1 + \frac{20}{s^2+16}\right)\right]_s^\infty$

$= \frac{1}{2}\left[0 + \log\left(\frac{s^2+16}{s^2+36}\right)\right]$

$\therefore L\left[\frac{\cos(6t) - \cos(4t)}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$

$\downarrow$

By First definition, $\rightarrow \int_0^\infty e^{-st} \cdot f(t) \cdot dt$

For $\int_0^\infty f(t) \cdot dt$, put $s = 0$

$\therefore \frac{1}{2}\log\left(\frac{0^2+16}{0^2+36}\right) = \log\left(\frac{4}{6}\right) = \log\left(\frac{2}{3}\right)$ //

- Laplace of derivative property :-

$$L[f^{(n)}(t)] = s^n \cdot L[f(t)] - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - s^{n-3} \cdot f''(0) - \dots - f^{(n-1)}(0)$$

$n=1$, $L[f'(t)] = s \cdot L[f(t)] - f(0)$

~~n=2~~

$n=2$, $L[f''(t)] = s^2 \cdot L[f(t)] - s \cdot f(0) - f'(0)$

$n=3$, $L[f'''(t)] = s^3 \cdot L[f(t)] - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$

Q. Using laplace transform of cos(at), find laplace transform of sin(at). [Derivative Property]

Ans. we know that $L[\cos(at)] = \frac{s}{s^2+a^2}$

but $-a\sin(at) = \frac{d}{dt}(\cos(at))$

$\therefore -\frac{1}{a} L\left[\frac{d}{dt}(\cos(at))\right] = L[\sin(at)]$

$\therefore L[\sin(at)] = -\frac{1}{a}\left(s \cdot L[f(t)] - f(0)\right)$

$\therefore$ when $f(t) = \cos(at)$

$= -\frac{1}{a}\left(s\left(\frac{s}{s^2+a^2}\right) - 1\right)$

$= -\frac{1}{a}\left(\frac{\cancel{s^2} - \cancel{s^2} - a^2}{s^2+a^2}\right)$

$\therefore L[\sin(at)] = \frac{a}{s^2+a^2}$ //

- **Laplace of integral property :**

If $L[f(t)] = \phi(s)$,
then $L\left[\int_0^t f(u)\cdot du\right] = \dfrac{\phi(s)}{s}$

Q. Find laplace transform of following functions:
(i) $erf_c(\sqrt{t})$
(ii) $\int_t^{\infty} \cos(u)/u \cdot du$
(iii) $\cosh(t)\cdot\int_0^t e^{u}\cosh(u)\cdot du$
(iv) $\int_0^t u\cdot e^{-3u}\cdot\cos^2(2u)\cdot du$

① $erf_c(\sqrt{t}) = 1 - erf(\sqrt{t})$ (is a property)★
$\therefore L[erf_c(\sqrt{t})] = L(1) - L[erf(\sqrt{t})]$
Now,
$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\cdot du$$ [Definition of erf (error function)]

Let $u^2 = v$ $\therefore 2u\,du = dv$ $du = dv/2\sqrt{v}$
$$\therefore erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-v}\cdot v^{-1/2}\,dv$$

$$\therefore L[erf(\sqrt{t})] = \frac{1}{\sqrt{\pi}}L\left[\int_0^t e^{-v}\cdot v^{-1/2}\,dv\right]$$
$$= \frac{1}{\sqrt{\pi}}\cdot\frac{\phi(s)}{s}$$ [where $\phi(s) = L[e^{-v}\cdot v^{-1/2}]$]
$$\therefore L[e^{-v}\cdot v^{-1/2}] = \frac{\sqrt{1/2}}{(s+1)^{1/2}}$$
$$\therefore L[erf(\sqrt{t})] = \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{\sqrt{s+1}}\cdot\frac{1}{s} = \frac{1}{s\sqrt{s+1}}$$
$$\therefore L[erf_c(\sqrt{t})] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{\sqrt{s+1}-1}{s\sqrt{s+1}}$$

② $\int_t^\infty \frac{\cos(u)}{u} \cdot du$

Let $f(t) = \int_t^\infty \frac{\cos u}{u} \cdot du$

$\therefore$ Put $u = vt$, $du = t \cdot dv$

$$\therefore f(t) = \int_{v=1}^\infty \frac{\cos(vt)}{vt} \cdot t\,dv$$

$$\therefore L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$= \int_{t=0}^\infty e^{-st} \left[\int_{v=1}^\infty \frac{\cos(vt)}{v} dv\right] dt$$

$$= \int_{v=1}^\infty \frac{dv}{v} \left[\int_{t=0}^\infty e^{-st} \cdot \cos(vt) \cdot dt\right]$$

$$= \int_1^\infty \frac{dv}{v} \cdot L[\cos(vt)] \qquad = \int_1^\infty \frac{dv}{v} \cdot \frac{s}{s^2+v^2}$$

$$= s \int_1^\infty \frac{1}{v(v^2+s^2)} dv$$

$$= \frac{1}{s} \int_1^\infty \left(\frac{1}{v} - \frac{v}{v^2+s^2}\right) dv \qquad \text{(by partial fractions)}$$

$$= \frac{1}{s} \left[\log(v) - \frac{1}{2}\log(v^2+s^2)\right]_1^\infty$$

$$= \frac{1}{2s} \left[\log\left(\frac{v^2}{v^2+s^2}\right)\right]_1^\infty \qquad = \frac{1}{2s}\left[\log\left(1+\frac{s^2}{v^2}\right)\right]_\infty^1$$

$$= \frac{1}{2s}\left[\log(1+s^2) - \log(1)\right]$$

$$= \frac{1}{2s} \cdot \log(1+s^2)$$

//

③ $\cosh(t) \cdot \int_0^t e^u \cdot \cosh(u) \cdot du$

$\rightarrow = L\left[\left(\frac{e^t + e^{-t}}{2}\right) \cdot \underbrace{\int_0^t e^u \cdot \cosh(u) du}_{f(t)}\right]$

$= \frac{1}{2} L\left[e^t f(t) + e^{-t} f(t)\right]$

$= \frac{1}{2} L\left[e^t \cdot f(t)\right] + \frac{1}{2} L\left[e^{-t} \cdot f(t)\right]$

$= \frac{1}{2}\left[\phi(s-1) + \phi(s+1)\right]$

where, $\phi(s) = L[f(t)] = L\left[\int_0^t e^u \cdot \cosh(u) du\right]$

~~Let $L[e^u \cdot \cosh(u)] = \phi(s)$~~ $= \frac{1}{s}\left[\frac{s-1}{(s-1)^2 - 1}\right]$

$\therefore$ ~~$\phi(s) = \frac{1}{s_1}$~~

$$\boxed{\begin{array}{l} \text{To find } L[e^u \cosh(u)] \\ L[\cosh(u)] = \frac{s}{s^2 - 1} \quad \therefore L[e^u \cosh(u)] = \frac{s-1}{(s-1)^2 - 1} \end{array}}$$

$\therefore$ original Laplace $= \frac{1}{2}\left[\frac{s-2}{(s-2)^2 - 1}\left(\frac{1}{(s-1)}\right) + \frac{s}{s^2 - 1} \cdot \left(\frac{1}{s+1}\right)\right]$

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(4) $\int_0^t u\cdot e^{-3u}\cdot\cos^2(2u)\,du$