Inverse Laplace Transform

* If $L[f(t)] = \phi(s)$

$$= \int_0^\infty e^{-st} \cdot f(t)\,dt$$

then f(t) is called inverse laplace transform of $\phi(s)$

$$f(t) = L^{-1}[\phi(s)]$$

- Formulae for Laplace Inverse :-

→ $L[1] = \frac{1}{s}$ $\quad \therefore L^{-1}\left[\frac{1}{s}\right] = 1$ //

→ $L[e^{at}] = \frac{1}{s-a}$ $\quad \therefore L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ //

Similarly, $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$ //

→ $L[\cos(at)] = \frac{s}{s^2+a^2}$ $\quad \therefore L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at)$ //

Similarly,

$L[\sin(at)] = \frac{a}{s^2+a^2}$ $\quad \therefore L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin(at)}{a}$ //

→ $L[\cosh(at)] = \frac{s}{s^2-a^2}$ $\quad \therefore L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh(at)$ //

Similarly,

$L[\sinh(at)] = \frac{a}{s^2-a^2}$ $\quad \therefore L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh(at)}{a}$ //

→ $L[t^n] = \dfrac{\lceil n+1}{s^{n+1}}$ $\therefore L^{-1}\left[\dfrac{1}{s^{n+1}}\right] = \dfrac{t^n}{\lceil n+1}$

OR $L^{-1}\left[\dfrac{1}{s^n}\right] = \dfrac{t^{n-1}}{\lceil n}$

- Linearity Property :

$$\boxed{L^{-1}[a\cdot\phi_1(s) + b\cdot\phi_2(s)] = a\,L^{-1}[\phi_1(s)] + b\,L^{-1}[\phi_2(s)]}$$

Q. Find $L^{-1}\left[\dfrac{2s-5}{4s^2+25} - \dfrac{(4s-18)}{9-s^2} + \dfrac{1}{s+1} + \dfrac{5}{s^4}\right]$

Ans. $= L^{-1}\left[\dfrac{2}{4}\left(\dfrac{s}{s^2+(5/2)^2}\right) - \dfrac{5}{4}\left(\dfrac{1}{s^2+(5/2)^2}\right) + 4\left(\dfrac{s}{s^2-9}\right) - 18\left(\dfrac{1}{s^2-9}\right) + \dfrac{1}{s+1} + \dfrac{5}{s^4}\right]$

$= \dfrac{1}{2}L^{-1}\left[\left(\dfrac{s}{s^2+(5/2)^2}\right)\right] - \dfrac{5}{4}L^{-1}\left[\dfrac{1}{s^2+(5/2)^2}\right] + 4L^{-1}\left[\dfrac{s}{s^2-9}\right] - 18L^{-1}\left[\dfrac{1}{s^2-9}\right] + L^{-1}\left[\dfrac{1}{s+1}\right] + 5L^{-1}\left[\dfrac{1}{s^4}\right]$

$= \dfrac{1}{2}\times\cos(5t/2) - \dfrac{\cancel{5}}{\cancel{4}_2}\times\dfrac{\cancel{2}}{\cancel{5}}\sin(5t/2) + 4\cosh(3t) - \dfrac{18}{3}\sinh(3t) + e^{-t} + \dfrac{5t^3}{\lceil 4}$

$= \dfrac{\cos(5t/2)}{2} - \dfrac{\sin(5t/2)}{2} + 4\cosh(3t) - 6\sinh(3t) + e^{-t} + \dfrac{5t^3}{6}$

- **First Shifting Property:**

We know that if,

$$L[f(t)] = \phi(s)$$

then $L[e^{\mp at} f(t)] = \phi(s \pm a)$

$\therefore$ If $f(t) = L^{-1}[\phi(s)]$

then $L^{-1}[\phi(s \pm a)] = e^{\mp at} f(t)$

$$= e^{\mp at} \cdot L^{-1}[\phi(s)]$$

Q. $L^{-1}\left[\underbrace{\dfrac{s+3}{(s+3)^2+9}}_{\phi(s+3)}\right]$

Ans. $= e^{-3t} \cdot L^{-1}\left[\dfrac{s}{s^2+9}\right]$

$= e^{-3t} \cdot \cos(3t)$

Q. Find inverse laplace of following functions:

(i) $\dfrac{s^2}{(s-1)^3}$

(ii) $\dfrac{s}{s^4+s^2+1}$

(iii) $\dfrac{s^2+16s-24}{s^4+20s^2+64}$

Q.1 $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

Ans. $= L^{-1}\left[\frac{(s-1)^2 + (2s-1)}{(s-1)^3}\right]$

$= L^{-1}\left[\frac{(s-1)^2 + 2(s-1) + 1}{(s-1)^3}\right]$

$= L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]$

~~$= e^t L^{-1}\left[\frac{1}{s}\right] + 2e^t L^{-1}\left[\right.$~~

$= e^t + 2\cdot e^t \cdot L^{-1}\left[\frac{1}{s^2}\right] + e^t \cdot L^{-1}\left[\frac{1}{s^3}\right]$

$= e^t + 2\cdot e^t \cdot \frac{t}{\Gamma 2} + e^t \cdot \frac{t^2}{\Gamma 3}$

$= e^t\left[1 + 2t + \frac{t^2}{2}\right]$ //

~~Q.2~~

Q.2 $\frac{s}{s^4+s^2+1}$

Ans. $= \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{1}{2}\left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right]$

$= \frac{1}{2}\left[\frac{1}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right] - \frac{1}{2}\left[\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right]$

∴ Taking Laplace Inverse,

$= \frac{1}{2} L^{-1}\left[\frac{1}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right]$

$= \frac{1}{2} \cdot e^{t/2} \cdot L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right] - \frac{1}{2} e^{-t/2} L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]$

$= \frac{1}{\cancel{2}} \cdot e^{t/2} \cdot \frac{\cancel{2}}{\sqrt{3}} \cdot \sin(\sqrt{3}t/2) - \frac{1}{\cancel{2}} \cdot e^{-t/2} \cdot \frac{\cancel{2}}{\sqrt{3}} \cdot \sin(\sqrt{3}t/2)$

$= \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \left[\frac{e^{t/2} - e^{-t/2}}{\sqrt{3}}\right]$

$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \left[\frac{e^{t/2} - e^{-t/2}}{2}\right]$

$= \frac{2}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \sinh\left(\frac{t}{2}\right)$ //

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Q.3 $\dfrac{s^2+16s-24}{s^4+20s^2+64}$

hints $\rightarrow$ $\left(\begin{array}{l} a=-4/3,\ b=10/3 \\ c=4/3,\ d=-7/3 \end{array}\right)$

hints $\rightarrow$ $\left[\dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+16}\right]$

Ans.

- **Second Shifting Property** :

We know that if,

$$L[f(t)] = \phi(s)$$

and $g(t) = \begin{cases} f(t-a) &, \text{for } t>a \\ 0 &, \text{for } t<a \end{cases}$

then $L[g(t)] = e^{-as}.\phi(s)$

$= e^{-as}.L[f(t)]$

$\therefore$ If $f(t) = L^{-1}[\phi(s)]$

then $L^{-1}[e^{-as}\phi(s)] = g(t)$

$= \begin{cases} f(t-a) &, \text{for } t>a \\ 0 &, \text{for } t<a \end{cases}$

Q. ~~For~~ Find $L^{-1}\left[\dfrac{e^{-2s}}{s^2+8s+25}\right]$

Ans. Let $\phi(s) = \dfrac{1}{s^2+8s+25} = \dfrac{1}{(s+4)^2+3^2}$

$\therefore L^{-1}[\phi(s)] = L^{-1}\left[\dfrac{1}{(s+4)^2+3^2}\right]$

$= e^{-4t}.L^{-1}\left[\dfrac{1}{s^2+3^2}\right]$

$= \dfrac{e^{-4t}.\sin(3t)}{3} = f(t)$

$\therefore L^{-1}[e^{-2s}.\phi(s)] = g(t)$ [2nd shifting property]

$= \begin{cases} f(t-2) &, \text{for } t>2 \\ 0 &, \text{for } t<2 \end{cases}$

$= \begin{cases} \dfrac{e^{-4(t-2)}.\sin(3(t-2))}{3} &, \text{for } t>2 \\ 0 &, \text{for } t<2 \end{cases}$

- Laplace Inverse of derivatives :

We know that if,

$L[f(t)] = \phi(s)$

then $L[t^n . f(t)] = (-1)^n . \frac{d^n}{ds^n} [\phi(s)]$

$\therefore$ If $f(t) = L^{-1}[\phi(s)]$

then $L^{-1}[\phi^{(n)}(s)] = (-1)^n . t^n . f(t)$

$= (-1)^n . t^n . \phi(s)$

Q. Find Laplace Inverse of following functions:

(i) $\frac{s+3}{(s^2+6s+10)^2}$

(ii) $\log\left(\frac{(s^2+1)}{s(s+1)}\right)$

(iii) $2\tanh^{-1}(s)$

(iv) $\tan^{-1}(2/s^2)$

Q.1 $\frac{s+3}{(s^2+6s+10)^2}$

Ans. Let $\phi(s) = \frac{1}{s^2+6s+10}$

$$\therefore \phi'(s) = \frac{-(2s+6)}{(s^2+6s+10)^2} = \frac{-2(s+3)}{(s^2+6s+10)^2}$$

$$\therefore L^{-1}[\phi'(s)] = -2 \cdot L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right]$$

$$= -t \cdot L^{-1}[\phi(s)]$$ (by ~~second~~ laplace inverse of derivat

$$\therefore L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right] = -\frac{1}{2} \times -t \times L^{-1}[\phi(s)]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s+3)^2+1^2}\right]$$

$$= \frac{t}{2} \cdot e^{-3t} \cdot L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \frac{t}{2} \cdot e^{-3t} \cdot \sin(t)$$

Q.2 $\log\left(\frac{s^2+1}{s(s+1)}\right)$

Ans. Let $\phi(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$

$$= \log(s^2+1) - \log(s) - \log(s+1)$$

$$\therefore \phi'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore L^{-1}[\phi'(s)] = 2\,L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

↓

by Laplace Inverse of Derivatives

↓

$$-t \cdot f(t) = 2 \cdot \cos(t) - 1 - e^{-t}$$

$$\therefore f(t) = \frac{-1}{t}\left[2\cos(t) - 1 - e^{-t}\right]$$

Q.3 $2\tanh^{-1}(s)$

Ans. Let $\phi(s) = 2\tanh^{-1}(s)$

$$= 2 \times \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$$

$$= \log(1+s) - \log(1-s)$$

☆ [Remember $\frac{d}{ds}\tanh^{-1}(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$]

$$\therefore \phi'(s) = \frac{1}{1+s} \mp \frac{1}{s-1}$$

$$\therefore L^{-1}[\phi'(s)] = L^{-1}\left[\frac{1}{s+1}\right] \mp L^{-1}\left[\frac{1}{s-1}\right]$$

↓

by Laplace Inverse of Derivatives

↓

$$-t \cdot L^{-1}[\phi(s)] = e^{-t} - e^{t}$$

$$\therefore L^{-1}[\phi(s)] = \frac{e^{t} - e^{-t}}{t}$$

Q.4 $\tan^{-1}(2/s^2)$

Ans. Let $\phi(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

$$\therefore \phi'(s) = \frac{1}{\left(\frac{2}{s^2}\right)^2+1} \cdot \frac{-4}{s^3}$$

$$= \frac{-4\cdot s^4}{(4+s^4)\cdot s^3} = \frac{-4s}{s^4+4}$$

$$\therefore L^{-1}[\phi'(s)] = -4\, L^{-1}\left[\frac{s}{(s^2+2)^2-4s^2}\right]$$

$$= -4\, L^{-1}\left[\frac{s}{(s^2-2s+2)(s^2+2s+2)}\right]$$

$$= -\, L^{-1}\left[\frac{4s}{(s^2-2s+2)(s^2+2s+2)}\right]$$

$$= -\cancel{4}\left[L^{-1}\left[\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right]\right]$$

$$= L^{-1}\left[\frac{1}{s^2+2s+2}\right] - L^{-1}\left[\frac{1}{s^2-2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] - L^{-1}\left[\frac{1}{(s-1)^2+1^2}\right]$$

$$= e^{-t}\cdot L^{-1}\left[\frac{1}{s^2+1^2}\right] - e^{t} L^{-1}\left[\frac{1}{s^2+1^2}\right]$$ (Using First Shifting property)

$$= e^{-t}\cdot \sin(t) - e^{t}(\sin(t))$$

by Laplace Inverse of Derivatives

$$-t\cdot L^{-1}[\phi(s)] = \sin(t)\left(e^{-t}-e^{t}\right)$$

$$\therefore L^{-1}[\phi(s)] = \sin(t)\left(\frac{e^{t}-e^{-t}}{t}\right)$$

- Convolution Theorem :

If $L[f_1(t)] = \phi_1(s)$
and $L[f_2(t)] = \phi_2(s)$

then $L^{-1}[\phi_1(s) \cdot \phi_2(s)] = \int_0^t f_1(u) \cdot f_2(t-u)\, du$

where, $f_1(t) = L^{-1}[\phi_1(s)]$ and $f_2(t) = L^{-1}[\phi_2(s)]$.

Q. Find inverse laplace using convolution theorem.

(i) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

(ii) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

★ Take Lengthy laplace function as $\phi_1(s)$ and simpler one as $\phi_2(s)$!!

Q.1 $\phi(s) = \dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

Ans. $= \dfrac{(s+1)^2+2}{((s+1)^2+1)((s+1)^2+4)}$

$\therefore L^{-1}[\phi(s)] = e^{-t} \cdot L^{-1}\left[\dfrac{s^2+2}{(s^2+1)(s^2+4)}\right]$ (by first shifting property)

$= e^{-t}\left[L^{-1}\left[\dfrac{s^2}{(s^2+1)(s^2+4)}\right] + \dfrac{2}{3} L^{-1}\left[\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right]\right]$

$= e^{-t}\left[L^{-1}\left[\dfrac{s^2}{(s^2+1)(s^2+4)}\right] + \dfrac{2}{3}\left(\sin(t) - \dfrac{1}{2}\sin(2t)\right)\right]$

Finding $L^{-1}\left(\dfrac{s^2}{(s^2+1)(s^2+4)}\right)$ using convolution,

Let $\phi_1(s) = \dfrac{s}{s^2+4}$ $\therefore L^{-1}[\phi_1(s)] = \cos(2t)$

$\phi_2(s) = \dfrac{s}{s^2+1}$ $\therefore L^{-1}[\phi_2(s)] = \cos(t)$

$\therefore L^{-1}[\phi_1(s)\,\phi_2(s)] = \int_0^t \cos(2u)\cdot\cos(t-u)\,du$

$= \dfrac{1}{2}\int_0^t \cos(u+t) + \cos(3u-t)\,du$

$= \dfrac{1}{2}\left[\sin(u+t) + \dfrac{\sin(3u-t)}{3}\right]_0^t$

Putting limits $\rightarrow = \dfrac{1}{2}\left[\dfrac{4}{3}\sin(2t) - \dfrac{2}{3}\sin(t)\right]$

$\therefore L^{-1}[\phi(s)] = e^{-t}\left[\left(\dfrac{2}{3}\sin(2t) - \dfrac{1}{3}\sin(t)\right) + \dfrac{2}{3}\sin(t) - \dfrac{1}{3}\sin(2t)\right]$

$= e^{-t}\left[\dfrac{1}{3}\sin(2t) + \dfrac{1}{3}\sin(t)\right]$

$L^{-1}[\phi(s)] = \dfrac{e^{-t}}{3}\left[\sin(2t) + \sin(t)\right]$

Q.2 $\phi(s) = \dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$

Ans. $= \dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

Let $\phi_1(s) = \dfrac{s+1}{(s+1)^2+1}$, $\phi_2(s) = \dfrac{s}{s^2+1}$

$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\dfrac{s+1}{(s+1)^2+1}\right]$

By first shifting property $\rightarrow$ $= e^{-t} . L^{-1}\left[\dfrac{s}{s^2+1}\right]$

$= e^{-u} . \cos(u)$

$L^{-1}[\phi_2(s)] = L^{-1}\left[\dfrac{s}{s^2+1}\right]$

$= \cos(u)$

By convolution theorem,

$L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u) f_2(t-u)\, du$

$= \int_0^t e^{-u} \cos(u) . \cos(t-u)\, du$

$= \frac{1}{2} \int_0^t e^{-u} \left(\cos(t) + \cos(2u-t)\right) du$

$= \frac{1}{2}\left[\int_0^t e^{-u} \cos(t)\, du + \int_0^t e^{-u} . \cos(2u-t)\, du\right]$

$= \frac{1}{2}\left[-\cos t\,(e^{-u}) + \dfrac{e^{-u}}{(1+4)} . \left(-\cos(2u-t) + 2\sin(2u-t)\right)\right]_0^t$

(next page)$\rightarrow$

Putting limits $= \frac{1}{2}\left[-\cos t\left(e^{-t}-1\right) + \frac{e^{-t}}{5}(-\cos t + 2\sin t) - \frac{1}{5}(-\cos t - 2\sin t)\right]$

$= \frac{1}{2}\left[\cos(t) - e^{-t}\cdot\cos(t) - \frac{e^{-t}\cdot\cos(t)}{5} + \frac{2e^{-t}\sin t}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5}\right]$

$= \frac{1}{2}\left[\cos(t)\left(\frac{6}{5} - \frac{6e^{-t}}{5}\right) + \sin(t)\left(\frac{2}{5} + \frac{2e^{-t}}{5}\right)\right]$

$= \frac{3}{5}\cos(t)\left[1-e^{-t}\right] + \frac{1}{5}\sin(t)\left[1+e^{-t}\right]$ //

Q. Find $L^{-1}\left[\frac{(s+3)^2}{(s^2+6s+5)^2}\right]$ using convolution theorem.

Ans $\therefore \phi(s) = \frac{(s+3)^2}{((s+3)^2-4)^2}$

By first shifting property.,

$$L^{-1}[\phi(s)] = e^{-3t}\left[\frac{s^2}{(s^2-2^2)^2}\right]$$

Let $\phi_1(s) = \phi_2(s) = \frac{s}{s^2-4}$

$\therefore L^{-1}[\phi_1(s)] = L^{-1}[\phi_2(s)] = \cosh(2u)$

$\therefore$ By convolution theorem.

$$L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u) f_2(t-u)\,du$$

$$= \int_0^t \cosh(2u)\cdot\cosh\cancel{(t-2u)}(2(t-u))\,du$$

$$= \frac{1}{2}\left[\int_0^t \cosh(2t)\,du + \int_0^t \cosh(4u-2t)\,du\right]$$

$$= \frac{1}{2}\left[\cosh(2t)\cdot[u]_0^t + \left[\frac{\sinh(4u-2t)}{4}\right]_0^t\right]$$

Putting limits $\rightarrow$ $= \frac{1}{2}\left[t\cdot\cosh(2t) + \left(\frac{\sinh(2t)}{4} + \frac{\sinh(2t)}{4}\right)\right]$

$$= \frac{t}{2}\cdot\cosh(2t) + \frac{1}{4}\cdot\sinh(2t)$$

//

Q. Find $\phi(s) = \frac{1}{s} \log\left(\frac{s+3}{s+2}\right)$

- Periodic function ↓

If $f(t) = f(t+a)$,

then $L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st}.f(t)\,dt$

Q. $f(t) = \begin{cases} k & \text{if } 0<t<a \\ -k & \text{if } a<t<2a \end{cases}$

Ans. $L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st}.f(t)\,dt$

$= \frac{1}{1-e^{-2as}} \left[\int_0^a k.e^{-st}dt + \int_a^{2a} -k.e^{-st}dt \right]$

$= \frac{k}{s(1-e^{-2as})} \left(-e^{-as} + 1 + e^{-2as} - e^{-as} \right)$

$= \frac{k}{s(1-e^{-2as})} (e^{-as} - 1)^2$

$\therefore \frac{k(e^{-as}-1)^2}{s(1-e^{-as})(1+e^{-as})} = \frac{+k}{s} \times \frac{\cancel{(e^{-as}-1)}(1-e^{-as})}{(1+e^{-as})}$

(~~Divide~~ Multiply by $e^{as/2}$ on both numerator and denominator)

$\therefore L[f(t)] = \frac{k}{s} \times \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}}$

$= \frac{k}{s} \times \frac{(e^{as/2} - e^{-as/2})/2}{(e^{as/2} + e^{-as/2})/2}$

$= \frac{k}{s} \times \frac{\sinh\left(\frac{as}{2}\right)}{\cosh\left(\frac{as}{2}\right)}$

$L[f(t)] = \frac{k}{s} \times \tanh\left(\frac{as}{2}\right)$ //

Q. Find Laplace transform of full wave rectification of

$f(t) = |\sin(wt)|$, $t \geq 0$

Ans. Period of function $= \dfrac{\pi}{w}$

$$\therefore L[f(t)] = \frac{1}{1-e^{-(\pi/w)s}} \int_0^{\pi/w} e^{-st}.\sin(wt)\,dt$$

$$= \frac{1}{1-e^{-\pi s/w}} \cdot \left[\frac{e^{-st}}{s^2+w^2} \left(-s\cdot\sin(wt) - w\cos(wt)\right) \right]_0^{\pi/w}$$

$$= \frac{1}{1-e^{-\pi s/w}} \left[\frac{e^{-\pi s/w}}{s^2+w^2} \left((-s\cdot 0) - w(-1)\right) - \frac{1}{s^2+w^2}(0-w) \right]$$

$=$

$$\therefore L[f(t)] = \frac{1}{1-e^{-\pi s/w}} \left(w \left(\frac{e^{-\pi s/w}}{(s^2+w^2)} + \frac{1}{(s^2+w^2)} \right) \right)$$

$$= \frac{w\cdot\left(1+e^{-\pi s/w}\right)}{(s^2+w^2)\cdot\left(1-e^{-\pi s/w}\right)}$$

- **Heavy Sides' Unit Step Function**

The function is defined by,

$$H(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

If $a=0$, then

$$H(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

→ $L[H(t-a)] = \dfrac{e^{-as}}{s}$

$\therefore L^{-1}\left[\dfrac{e^{-as}}{s}\right] = H(t-a)$

At $a=0$,

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$

$$\therefore H(t) = 1$$

★ • Important Result ↓

If $L[f(t)] = \phi(s)$,

then $L[f(t-a) \cdot H(t-a)] = e^{-as} \cdot \phi(s)$

ALSO ↓

If $f(t) = \begin{cases} f_1(t) & \text{for } 0 < t < a \\ f_2(t) & \text{for } a < t < b \\ f_3(t) & \text{for } t > b \end{cases}$

then,

$$f(t) = f_1(t)[H(t) - H(t-a)] + f_2(t)[H(t-a) - H(t-b)] + f_3(t)[H(t-b)]$$

Q. Find $L\left[\sin(t)\left(H(t-\pi/2) - H(t-3\pi/2)\right)\right]$

Ans. $= \int_{\pi/2}^{3\pi/2} e^{-st} \cdot \sin(t)\, dt$

$= \left[\frac{e^{-st}}{s^2+1}\left(-s \cdot \sin(t) - \cos(t)\right)\right]_{\pi/2}^{3\pi/2}$

$= \left(\frac{e^{-3s\pi/2}}{s^2+1}\right)(s) - \left(\frac{e^{-\pi s/2}}{s^2+1}\right)(-s)$

$= \frac{s}{s^2+1} \times \left(e^{-3\pi s/2} + e^{-\pi s/2}\right)$ //

Q. Express the following function in terms of unit step function and obtain its Laplace Transform.

~~Ans~~. $f(t) = \begin{cases} t^2 & , 0 < t < 1 \\ 4t & , t > 1 \end{cases}$

Ans. $\therefore f(t) = t^2 \cdot [H(t) - H(t-1)] + 4t\,[H(t-1)]$

$= t^2 \cdot H(t) + (4t - t^2) \cdot H(t-1)$

$\therefore L[f(t)] = L[t^2 \cdot H(t)] + L[(4t - t^2) \cdot H(t-1)]$

We know,

$$L[f(t) \cdot H(t-a)] = e^{-as} \cdot L[f(t+a)]$$

$\therefore L[f(t)] = L[t^2] + e^{-s} \cdot L[4(t+1) - (t+1)^2]$

$= \frac{\lfloor 2+1}{s^{2+1}} + e^{-s} \cdot L[2t + 3 - t^2]$

$\therefore L[f(t)] = \frac{2}{s^3} + e^{-s}\left(\frac{2}{s^2} + \frac{3}{s} - \frac{2}{s^3}\right)$

- **Dirac Delta Function:** (Unit Impulse Function)

Consider $F(t) = \begin{cases} \frac{1}{\epsilon} & \text{, for } a < t < a+\epsilon \\ 0 & \text{, otherwise} \end{cases}$

$\therefore \lim_{\epsilon \to 0} F(t) = \begin{cases} \infty & \text{, at } t = a \\ 0 & \text{, otherwise} \end{cases}$

$= \delta(t-a)$

$$\left[\int_0^\infty \delta(t-a)\, dt = 1 \right]$$

→ $L[\delta(t-a)] = e^{-as}$

If $a = 0$,

then $L[\delta(t)] = 1 \Rightarrow L^{-1}[1] = \delta(t)$

→ $L[f(t) \cdot \delta(t-a)] = e^{-as} \cdot f(a)$

Q. Find $L\left[t \cdot H(t-a) + t^2 \cdot \delta(t-a)\right]$

Ans. $= L\left[\underbrace{t}_{f(t)} \cdot H(t-a)\right] + L\left[\underbrace{t^2}_{f(t)} \cdot \delta(t-a)\right]$

$= e^{-as} \cdot L\left[(t+a)\right] + e^{-as}\left[a^2\right]$

$= e^{-as} \cdot \left(\frac{1}{s^2} + \frac{a}{s}\right) + e^{-as} \cdot a^2$

Q. Evaluate : $\int_0^\infty t \cdot e^{2t} \cdot \sin(3t) \cdot \delta(t-2)\, dt$

Ans. $\therefore \int_0^\infty \underbrace{e^{-(-2t)} \cdot t \cdot \sin(3t) \cdot \delta(t-2)}_{f(t)}\, dt$

$= L\left[f(t)\right]_{s=-2}$

$\therefore L\left[\underbrace{t \cdot \sin(3t)}_{f_1(t)} \cdot \delta(t-2)\right]_{s=-2}$

We know.

$L\left[f(t) \cdot \delta(t-a)\right] = e^{-as} \cdot f(a)$

$\therefore L\left[f_1(t) \cdot \delta(t-2)\right]_{\substack{s=-2 \\ a=+2}} = e^{-2s} \cdot f_1(2)$

$= e^{-2s} \cdot 2 \cdot \sin(6)$

$\cancel{e^4} = 2 \cdot e^4 \cdot \sin(6)$ //

- Laplace for solving Differential Equations :

① $L[f'(t)] = s \cdot L[f(t)] - f(0)$

② $L[f''(t)] = s^2 \cdot L[f(t)] - s \cdot f(0) - f'(0)$

③ $L[f'''(t)] = s^3 \cdot L[f(t)] - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$

Q. Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3 \cdot t \cdot e^{-t}$

given $y(0) = 4$ and $y'(0) = 2$

Ans. $L[y''] + 2L[y'] + L[y] = 3L[t \cdot e^{-t}]$

$= [s^2 \cdot \bar{y} - s \cdot y(0) - y'(0)] + 2[s \cdot \bar{y} - y(0)] + [\bar{y}] = \frac{3}{(s+1)^2}$

~~By~~ But $y(0) = 4$ and $y'(0) = 2$.

$= [s^2 \cdot \bar{y} - 4s - 2] + 2[s \cdot \bar{y} - 4] + \bar{y} = \frac{3}{(s+1)^2}$

$= \bar{y}(s^2 + 2s + 1) - 4s - 10 = \frac{3}{(s+1)^2}$

$\therefore \bar{y} = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2}$

Breaking further → $\bar{y} = \frac{3}{(s+1)^4} + \frac{4}{s+1} + \frac{6}{(s+1)^2}$

$$\therefore y = f(t) = L^{-1}\left[\frac{3}{(s+1)^4} + \frac{6}{(s+1)^2} + \frac{4}{s+1}\right]$$

$$= 3\cdot e^{-t} L^{-1}\left[\frac{1}{s^4}\right] + 6\cdot e^{-t}\left[\frac{1}{s^2}\right] + 4\cdot e^{-t}\left[\frac{1}{s}\right]$$

$$= \frac{\cancel{3}\cdot e^{-t}\cdot t^3}{2} + \cancel{3}6e^{-t}\cdot t + 4\cdot e^{-t}(1)$$

Q. Solve $y'' + y = t$

Given : $y(\pi) = 0$ and $y'(0) = 1$

Ans. Let $y(0) = \alpha$.

$\therefore$ Taking Laplace on both sides,

$L[y''] + L[y] = L[t]$

Using formulae,

$$[s^2\bar{y} - s\,y(0) - y'(0)] + \bar{y} = \frac{1}{s^2}$$

$y(0) = \alpha$, $y'(0) = 1$

$$\therefore s^2\bar{y} - \alpha s - 1 + \bar{y} = \frac{1}{s^2}$$

$$\bar{y}(s^2+1) = \frac{1}{s^2} + \alpha s + 1$$

$$\bar{y} = \frac{1}{s^2(s^2+1)} + \frac{\alpha s + 1}{s^2+1} = \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + \frac{\alpha s}{s^2+1} + \frac{1}{s^2+1}$$

$$L[f(t)] = \frac{1}{s^2} - \cancel{\frac{1}{s^2+1}} + \frac{\alpha s}{s^2+1} + \cancel{\frac{1}{s^2+1}}$$

$$\therefore f(t) = L^{-1}\left(\frac{1}{s^2}\right) + \alpha L^{-1}\left(\frac{s}{s^2+1}\right)$$

$f(t) = t + \alpha\cdot\cos(t)$ [but $y(\pi) = 0$]

$\therefore f(\pi) = \pi + \alpha\cdot(-1)$ $\therefore \alpha = \pi$

$$f(t) = t + \pi\cdot\cos(t)$$

Q. $y'' + 2y' + 5y = 8\sin(t) + 4\cos(t)$

Given $y(0) = 1$ and $y(\pi/4) = \sqrt{2}$

Ans. Let $y'(0) = \alpha$.

Taking Laplace on both sides,

$$L[y''] + 2L[y'] + 5L[y] = 8L[\sin(t)] + 4L[\cos(t)]$$

$$\therefore [s^2\bar{y} - s\cdot y(0) - y'(0)] + 2[s\cdot\bar{y} - y(0)] + 5\bar{y} = \frac{8}{s^2+1} + \frac{4s}{s^2+1}$$

$$(s^2+2s+5)\bar{y} - s - \alpha - 2 = \frac{4s+8}{s^2+1}$$

$$\therefore \bar{y} = \frac{4s+8}{(s^2+2s+5)(s^2+1)} + \frac{s+\alpha+2}{s^2+2s+5}$$

$$\bar{y} = 2\left(\frac{1}{s^2+1} - \frac{1}{s^2+2s+5}\right) + \frac{(s+1)+(\alpha+1)}{(s+1)^2+4}$$

$$\therefore f(t) = 2L^{-1}\left(\frac{1}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+2s+5}\right) + L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) + (\alpha+1)L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= 2\sin(t) - \frac{2e^{-t}}{2}\sin(2t) + e^{-t}\cos(2t) + \frac{(\alpha+1)e^{-t}}{2}\sin(2t)$$

$$f(t) = 2\sin(t) + e^{-t}\cos(2t) + \frac{(\alpha-1)}{2}e^{-t}\sin(2t)$$

By $y(\pi/4) = \sqrt{2}$

$$\therefore \sqrt{2} = \sqrt{2} + 0 + \left(\frac{\alpha-1}{2}\right)e^{-\pi/4} \qquad \therefore \alpha = 1$$

$$\therefore f(t) = 2\sin(t) + e^{-t}\cos(2t)$$

Q. $\dfrac{dy}{dx} + 2y + \int_0^t y\,dt = \sin(t)$

Given $y(0) = 1$

Ans: $y(t) = -\frac{3}{2}e^{-t}\cdot t + \frac{1}{2}\sin(t) + e^{-t}$

Ans. $\therefore \cancel{y' + 2y + \left[\frac{y^2}{2}\right]_0^t = \sin(t)}$

$\cancel{y' + 2y = \sin(t) - \frac{t^2}{2}}$

$$L[y'] + 2L[y] + L\left[\int_0^t y\,dt\right] = \frac{1}{s^2+1}$$

$$\therefore \left[s\cdot\bar{y} - y(0)\right] + 2\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s^2+1}$$

$$\therefore \left(s+2+\frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1}$$

$$\left(\frac{s^2+2s+1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1$$

$$\therefore \bar{y} = \frac{s}{(s^2+1)(s+1)^2} + \frac{s}{(s+1)^2}$$

Q. Solve $y(t) = kt + \int_0^t y(u) \cdot \sin(t-u)\,du$

Ans. By convolution,

$$L\left[\int_0^t f_1(u) f_2(t-u)\,du\right] = \phi_1(s) \cdot \phi_2(s)$$
$$= L[f_1(t)] \cdot L[f_2(t)]$$

$\therefore$ Taking Laplace on both sides,

$$\bar{y} = \frac{k}{s^2} + \frac{\bar{y}}{s^2+1}$$

$$\bar{y}\left(1 - \frac{1}{s^2+1}\right) = \frac{k}{s^2} \qquad \therefore \bar{y} = \frac{k(s^2+1)}{s^4}$$

$$\bar{y} = \frac{k}{s^2} + \frac{k}{s^4}$$

$$\therefore f(t) = \frac{k \times t^{2-1}}{\Gamma 2} + \frac{k \times t^{4-1}}{\Gamma 4}$$

$$f(t) = kt + \frac{kt^3}{6}$$

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Q. Solve $\frac{d^2y}{dt^2} + 4y = g(t)$, given $y(0)=0$ and $y'(0)=1$

and $g(t) = \begin{cases} 1 & \text{for } 0<t<1 \\ 0 & \text{for } t>1 \end{cases}$

Ans:

$$y(t) = \frac{1}{2}\sin(2t) + \frac{(1-\cos(2t))}{4} - \frac{[1-\cos(2(t-1))]}{4} \cdot u(t-1)$$

Ans.