- **Fourier Integral**

If $f(x)$ satisfies Dirichlet's conditions in each finite interval $-L \le x \le L$ and if $f(x)$ is integrable in the interval $(-\infty, \infty)$ then Fourier Integral Theorem states that:

$$\boxed{f(x) = \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \cos(w(s-x))\, dw\, ds} \leftarrow \text{[Fourier Integral]}$$

→ **Fourier Sine and Cosine Integral**

Equation of Fourier Integral can be written as,

$$f(x) = \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \cos(ws) \cos(wx)\, dw\, ds$$
$$+ \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \sin(ws) \sin(wx)\, dw\, ds$$

- **Fourier Cosine Integral**

When $f(x)$ is an even function,

then $f(s)$ is even.

then $f(s) \sin(ws)$ is odd

★ and $f(s) \cdot \cos(ws)$ is even

$$\therefore \boxed{f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \cos(wx) \left[ \int_{s=0}^{\infty} f(s) \cos(ws)\, ds \right] dw} \leftarrow \text{[Fourier Cosine Integral (even)]}$$

- **Fourier Sine Integral**

If $f(x)$ is odd, then $f(s)$ is odd

★ then $f(s) \cdot \sin(ws)$ is even, and $f(s) \cdot \cos(ws)$ is odd.

$$\therefore \boxed{f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \sin(wx) \left[ \int_{s=0}^{\infty} f(s) \sin(ws)\, ds \right] dw} \leftarrow \text{[Fourier Sine Integral (odd)]}$$

Q. Express the following function $f(x) = \begin{cases} 1, & \text{for } |x|<1 \\ 0, & \text{for } |x|>1 \end{cases}$ as Fourier Integral. Hence evaluate $\int_0^\infty \frac{\sin(w)\cos(wx)}{w}\,dw$

Ans. Clearly, $f(x)$ is even, $\therefore f(s)$ is even.

$$\therefore f(x) = \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx) \int_{s=0}^{\infty} f(s)\cos(ws)\,ds\,dw$$

$$= \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx)\left[\int_{s=0}^{1} (1)\cos(ws)\,ds\right]dw \quad \left(\text{since } 1 \le x \le \infty,\ f(x)=0\right)$$

$$= \frac{2}{\pi}\int_{w=0}^{\infty} \cos(wx)\left[\frac{\sin(ws)}{w}\right]_0^1 dw$$

$$f(x) = \frac{2}{\pi}\int_{w=0}^{\infty} \frac{\cos(wx)\cdot\sin(w)}{w}\,dw$$

$$\therefore \frac{\pi}{2}\cdot f(x) = \int_0^\infty \frac{\sin(w)\cos(wx)}{w}\,dw$$

$$\therefore \int_0^\infty \frac{\sin(w)\cos(wx)}{w}\,dw = \begin{cases} \frac{\pi}{2}, & \text{for } |x|<1 \\ 0, & \text{for } |x|>1 \\ \pi/4, & \text{for } |x|=1 \end{cases}$$

At $|x| = 1$, (i.e. at $x = \pm 1$)

$f(x)$ is discontinuous.

$$\therefore f(1) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+}\right]$$

$$= \frac{1}{2}(1+0)$$

$$\therefore f(1) = 1/2 = f(-1)$$

Remember!

$$\int_0^\infty \frac{\sin(w)}{w}\,dw = \frac{\pi}{2}$$

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H.W.

Q. Find Fourier Integral Representation for ~~f(x)~~

$$f(x) = \begin{cases} 1-x^2, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$$

Q. Find Fourier Sine Integral representation for $f(x) = e^{-ax}/x$

Ans: $f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \sin(wx) \left[ \int_{s=0}^{\infty} f(s) \sin(ws)\, ds \right] dw$ ← Fourier sine Integral

Let $I(w) = \int_{s=0}^{\infty} \frac{e^{-as}}{s} \cdot \sin(ws)\, ds$ * [using DUIS concept ahead] (can even use Laplace)

$\therefore \frac{dI}{dw} = \frac{d}{dw} \int_{s=0}^{\infty} \frac{e^{-as}}{s} \cdot \sin(ws)\, ds$

$= \int_{s=0}^{\infty} \frac{\partial}{\partial w} \left( \frac{e^{-as}}{s} \cdot \sin(ws) \right) ds$

$= \int_{s=0}^{\infty} e^{-as} \cdot \cos(ws)\, ds$

$= \left[ \frac{e^{-as}}{a^2+w^2} \cdot \left( -a\cos(ws) + w\sin(ws) \right) \right]_0^{\infty}$

$\frac{dI}{dw} = \frac{a}{a^2+w^2}$

$\therefore I(w) = \int \frac{a}{a^2+w^2}\, dw$

$I(w) = \frac{a}{a} \tan^{-1}\left(\frac{w}{a}\right) + c \quad = \tan^{-1}\left(\frac{w}{a}\right) + c$ //

At $w=0$, $I(w)=0$ $\quad \therefore 0 = \tan^{-1}(0) + c$

$\therefore c = 0$ //

$\therefore I(w) = \tan^{-1}\left(\frac{w}{a}\right)$

$\therefore f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \sin(wx) \tan^{-1}\left(\frac{w}{a}\right) dw$ //

- Fourier Transform ↓

Fourier Transform for the function $f(t)$ is given by:

$$\boxed{F[f(t)] = \int_{-\infty}^{\infty} f(t) \cdot e^{ist} \, dt} \leftarrow \text{[Fourier Transform]}$$

and Inverse Fourier Transform of $F[f(t)]$ is given by:

$$\cancel{f(t)} = \boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F[f(t)] \, e^{-isx} \, ds} \leftarrow \text{[Inverse Fourier Transform]}$$

→ Fourier Sine Transform ↓

$$\boxed{F_s[f(t)] = \int_{0}^{\infty} f(t) \cdot \sin(st) \, dt} \leftarrow \text{[Fourier Sine Transform]}$$

$$\boxed{f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s[f(t)] \sin(sx) \, ds} \leftarrow \text{[Inverse ~~Fourier~~ Fourier Sine Transform]}$$

- Fourier Cosine Transform ↓

$$\boxed{F_c[f(t)] = \int_{0}^{\infty} f(t) . \cos(st) \, dt} \leftarrow \text{[Fourier Cosine Transform]}$$

$$\boxed{f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c[f(t)] . \cos(sx) \, ds} \leftarrow \text{[Inverse Fourier Cosine Transform]}$$

- Modulation Theorem :

$\rightarrow F[f(x) \cdot \cos(ax)] = \frac{1}{2}[F(s+a) + F(s-a)]$ $\quad (F(s) = F[f(t)])$

$\rightarrow F_s[f(t) \cdot \cos(at)] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

$\rightarrow F_c[f(t) \cdot \sin(at)] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

$\rightarrow F_s[f(t) \cdot \sin(at)] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

$\rightarrow F_c[f(t) \cos at] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$.

Q. $f(x) = \begin{cases} 1, & \text{for } |x| < 1 \\ 0, & \text{for } |x| > 1 \end{cases}$ Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$

Ans. $F[f(x)] = \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$

$= \int_{-1}^{1} e^{isx} dx = \left[\frac{e^{isx}}{is}\right]_{-1}^{1}$

$= \cancel{e^{isx} - e^{-isx}}$

$F[f(x)] = \frac{e^{is} - e^{-is}}{is} = F(s)$

$\therefore F(s) = \frac{2i\sin(s)}{is} \begin{cases} = \frac{2\sin(s)}{s}, & \text{if } s \neq 0 \\ = 2 & \text{if } s = 0 \end{cases}$ (using limits)

Now, taking Inverse Fourier Transform,

$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(s)}{s} e^{-isx} ds = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Putting $x = 0$,

$\therefore f(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(s)}{s} ds = \frac{2}{\pi} \int_0^\infty \frac{\sin(s)}{s} ds$

$\therefore \int_0^\infty \frac{\sin(s)}{s} ds = \frac{\pi}{2}$

Q. Find Fourier Transform of $f(x) = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$.

Hence evaluate $\displaystyle\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx$

Ans. $F[f(x)] = \displaystyle\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\, dx = F(s)$

$$= \int_{-1}^{1} (1-x^2)\cdot e^{isx}\, dx$$

$$\therefore F[f(x)] = \left[ \cancel{(1-x^2)\frac{e^{isx}}{is}}^{\,0} - (-2x)\frac{e^{isx}}{i^2s^2} + (-2)\frac{e^{isx}}{i^3s^3}\right]_{-1}^{1}$$

$$= \left[\left(\frac{2e^{is}}{-s^2} - \frac{2e^{is}}{-is^3}\right) - \left(\frac{-2e^{-is}}{-s^2} - \frac{2e^{-is}}{-is^3}\right)\right]$$

$$= \frac{-2}{s^2}\left(e^{is}+e^{-is}\right) + \frac{2}{s^3}\left(\frac{e^{+is}-e^{-is}}{i}\right)$$

$$= \frac{-4}{s^2}\left(\frac{e^{is}+e^{-is}}{2}\right) + \frac{4}{s^3}\left(\frac{e^{is}-e^{-is}}{2i}\right)$$

$$= \frac{-4}{s^2}(\cos(s)) + \frac{4}{s^3}(\sin(s))$$

$$F(s) = \frac{-4}{s^3}\left(s\cos(s) - \sin(s)\right) \quad //$$

Taking Inverse Fourier Transform,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}\left(s\cos(s)-\sin(s)\right) e^{-isx}\, ds$$

$$\therefore f(1/2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -4/s^3 \left(s\cos(s)-\sin(s)\right)\left(\cos\left(\tfrac{s}{2}\right) - i\sin\left(\tfrac{s}{2}\right)\right) ds$$

$$\therefore 3/4 + 0i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{-4}{s^3}\left(s\cos(s)-\sin(s)\right)\cdot\cos\left(\tfrac{s}{2}\right)ds}_{\text{even}} + 0i(0)$$

$$\therefore \frac{-4}{\pi}\int_0^\infty \frac{s\cos(s)-\sin(s)}{s^3}\cdot \cos\left(\tfrac{s}{2}\right) ds = \frac{3}{4} \quad \text{(Putting } s=x\text{)},$$

$$\therefore \int_0^\infty \left(\frac{x\cos(x)-\sin(x)}{x^3}\right)\cdot\cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16} \quad //$$

Q. Find Fourier Sine Transform of $f(x) = e^{-|x|}$, and show that

$$\int_0^\infty \frac{x \sin(mx)}{1+x^2}\,dx = \frac{\pi \cdot e^{-m}}{2}, \quad m>0.$$

Ans. $F_s(s) = F_s[f(x)] = \int_0^\infty f(x) \cdot \sin(sx)\,dx$

$$= \int_0^\infty e^{-x} \sin(sx)\,dx \leftarrow \begin{bmatrix} \text{since } 0 \text{ to } \infty = +ve \\ \text{but } e^{-|x|} \text{ means } -ve \end{bmatrix}$$

$$F_s[f(x)] = \left[\frac{e^{-x}}{1+s^2}\left(-\sin(sx) - s\cos(sx)\right)\right]_0^\infty$$

$$= \left[0 + \frac{s}{s^2+1}\right] \quad \therefore F_s[f(x)] = \frac{s}{s^2+1}$$

Using Inverse Fourier Sine Transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s[f(x)] \cdot \sin(sx)\,ds$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+1} \cdot \sin(sx)\,ds$$

(Replacing x by m, and s by x.)

$$\therefore e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x}{x^2+1} \cdot \sin(mx)\,dx$$

$$\therefore \int_0^\infty \frac{x \sin(mx)\,dx}{1+x^2} = \frac{\pi \cdot e^{-m}}{2}$$

Q. Find Fourier ~~Sine~~ Transform of $f(x) = e^{-x^2/2}$

Ans. $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{isx} dx$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-x^2/2} \cdot \cos(sx)}_{\text{even}} dx + i \int_{-\infty}^{\infty} \underbrace{e^{-x^2/2} \cdot \sin(sx)}_{\text{zero}} dx$$

$$\therefore I(s) = 2 \int_{0}^{\infty} e^{-x^2/2} \cdot \cos(sx) dx \quad \text{—— (1)}$$

$$\frac{dI}{ds} = 2 \int_{0}^{\infty} e^{-x^2/2} \cdot (-x \sin(sx)) dx = 2 \int_{0}^{\infty} (\sin(sx)) \cdot (-x \cdot e^{-x^2/2}) dx$$

we know, $\int e^{f(x)} \cdot f'(x) = e^{f(x)} + c$,

$$\therefore \frac{dI}{ds} = 2 \left[ \underbrace{\sin(sx) \int -x \cdot e^{-x^2/2} \cdot dx}_{\text{zero}} - \int_{0}^{\infty} \left( s\cos(sx) \cdot \int -x \cdot e^{-x^2/2} dx \right) dx \right]_{0}^{\infty}$$

$$= -s \left[ 2 \int_{0}^{\infty} \cos(sx) \cdot e^{-x^2/2} dx \right] = -sI$$

$$\therefore \frac{dI}{I} = -s\, ds \qquad \therefore \int \frac{dI}{I} = -\int s\, ds$$

$$\therefore \log(I) = -s^2/2 + c, \qquad \therefore I(s) = e^{-s^2/2} \cdot c$$

Putting $s=0$ in (1),

$$I(0) = \int_{0}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} \longrightarrow \text{(Remember!)}$$

$$\therefore I(0) = c \cdot e^{0} \qquad \therefore c = \sqrt{2\pi}$$

$$\therefore I(s) = \sqrt{2\pi} \cdot e^{-s^2/2}$$