"Discrete Mathematics and its Applications" Kenneth Rosen, 5th Edition, McGraw Hill.

Graph Theory

Chapter 8

Varying Applications (examples)

- Computer networks
- Distinguish between two chemical compounds with the same molecular formula but different structures
- Solve shortest path problems between cities
- Scheduling exams and assign channels to television stations

Topics Covered

- Definitions
- Types
- Terminology
- Representation
- Sub-graphs
- Connectivity
- Hamilton and Euler definitions
- Isomorphism of Graphs
- Planar Graphs

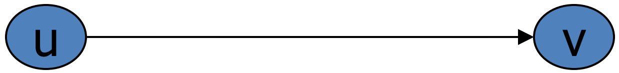
Definitions - Graph

A generalization of the simple concept of a set of dots, links, edges or arcs.

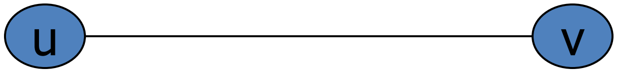
Representation: Graph G =(V, E) consists set of vertices denoted by V, or by V(G) and set of edges E, or E(G)

Definitions – Edge Type

Directed: Ordered pair of vertices. Represented as (u, v) directed from vertex u to v.



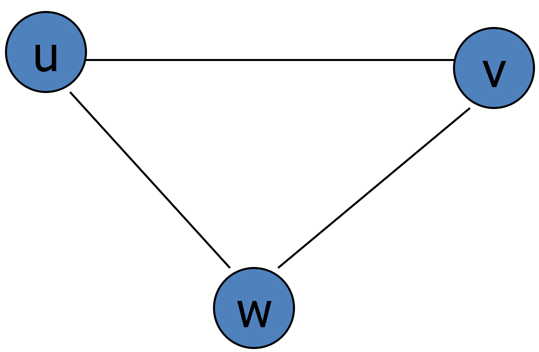
Undirected: Unordered pair of vertices. Represented as {u, v}. Disregards any sense of direction and treats both end vertices interchangeably.



Definitions – Graph Type

Simple (Undirected) Graph: consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges (undirected)

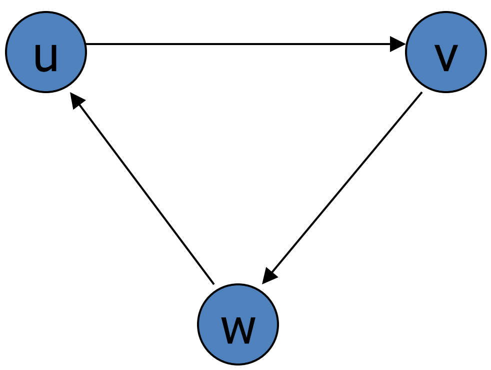
Representation Example: G(V, E), V = {u, v, w}, E = {{u, v}, {v, w}, {u, w}}



Definitions – Graph Type

Directed Graph: G(V, E), set of vertices V, and set of Edges E, that are ordered pair of elements of V (directed edges)

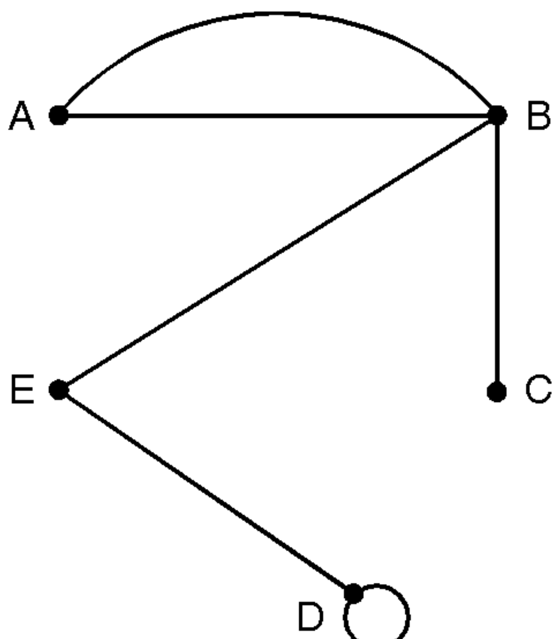
Representation Example: G(V, E), V = {u, v, w}, E = {(u, v), (v, w), (w, u)}

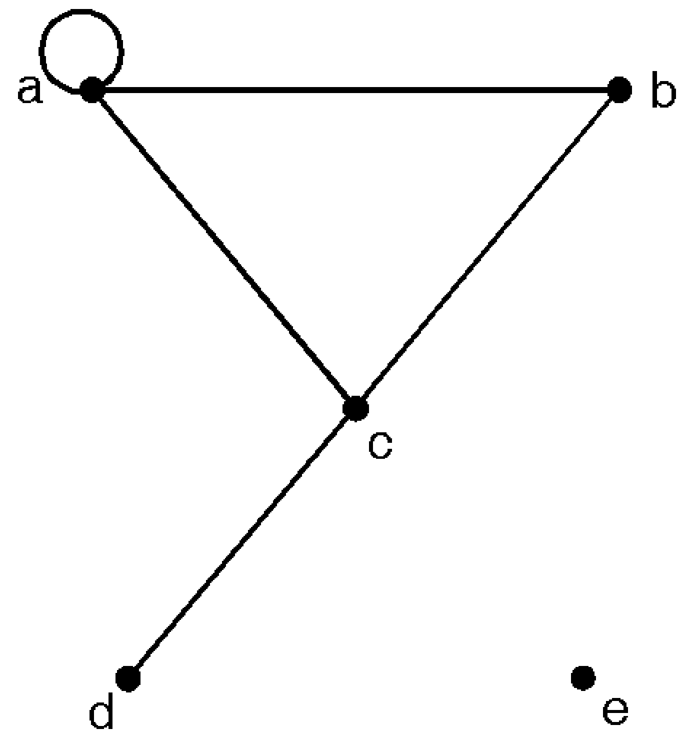


- **Degree of a vertex**: Number of edges having that vertex as an end point
- **Loop**: A graph may contain an edge from a vertex to itself referred to as a loop

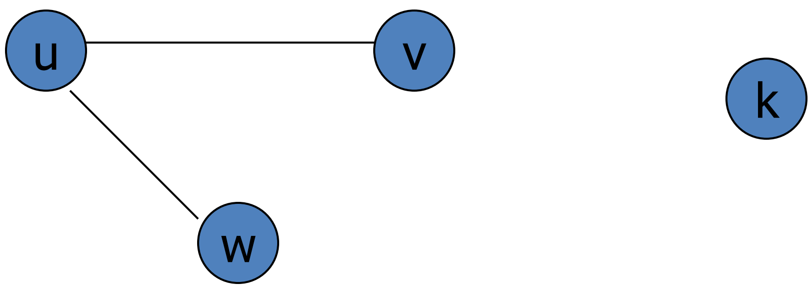
Isolated vertex: Vertex with degree 0

- **Adjacent vertices** : A pair of vertices that determine an edge





- For V = {u, v, w} ,
  E = { {u, w}, {u, w}, (u, v) },
  deg (u) = 2, deg (v) = 1, deg (w) = 1, deg (k) = 0,
- **k is isolated**



Terminology – Directed graphs

- **In-degree (u):** number of in coming edges

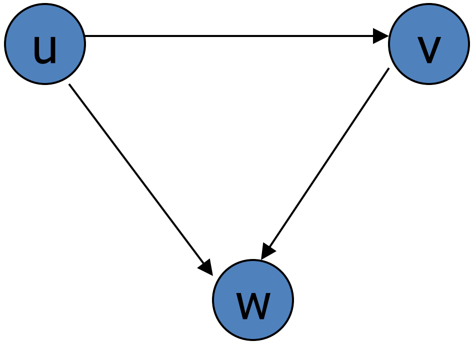
- **Out-degree (u):** number of outgoing edges

Representation Example: For V = {u, v, w} , E = { (u, w), (v, w), (u, v) },

indeg(u) = 0, outdeg (u) = 2,

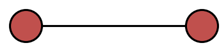
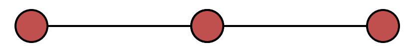
indeg(v) = 1, outdeg(v)= 1

indeg(w) = 2, outdeg (w) = 0



Types of Graphs

L2 L3

- Linear Graph  
- Discrete Graph (only vertices , no edges)

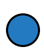
D2

D4

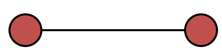
- Complete Graph
- Connected Graph

COMPLETE GRAPH

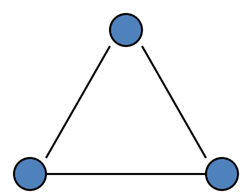
- **Complete graph:** K_n, where <u>every vertex is connected to every other vertex</u>
- **K_n is called a complete graph for n vertices if the number of edges are $n(n-1)/2$**
- DRAW COMPLETE GRAPH K_6



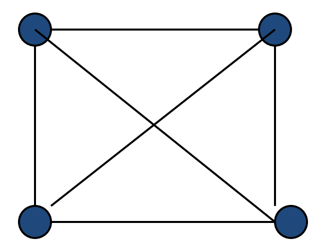
K_1



K_2

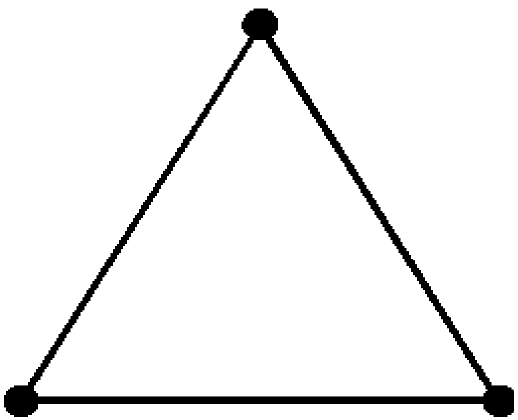


K_3

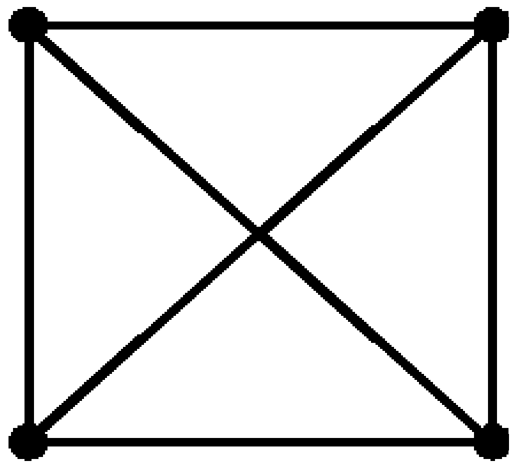


K_4

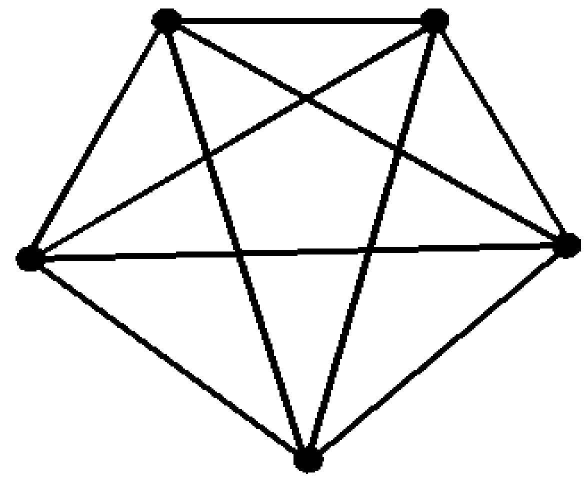
Representation Example: K_1, K_2, K_3, K_4



K_3



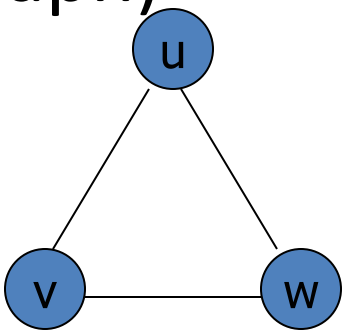
K_4

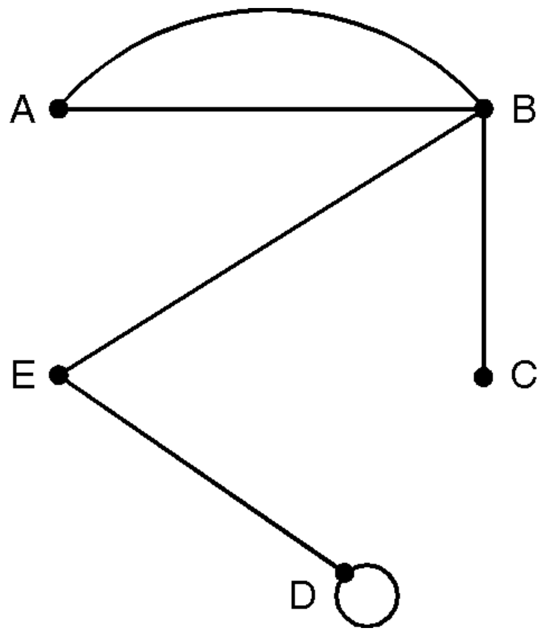


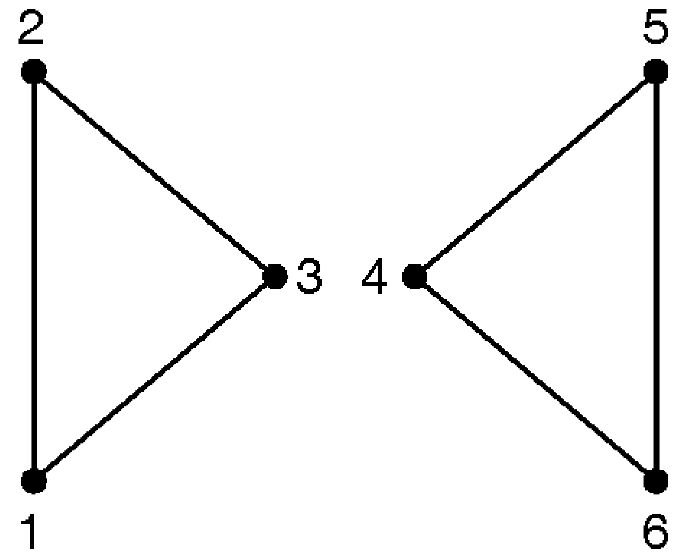
K_5

CONNECTED GRAPH

- If there is a path from any vertex to any other vertex in the graph
- Otherwise it is a disconnected graph(various connected pieces are called components of graph)

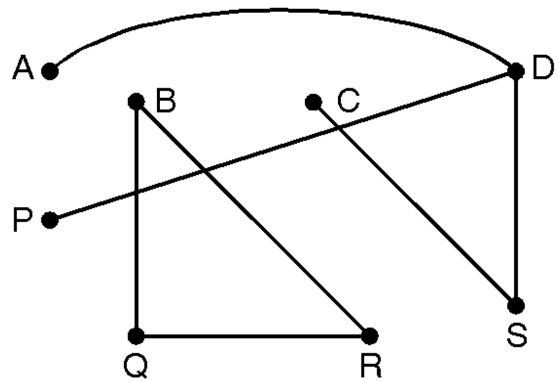




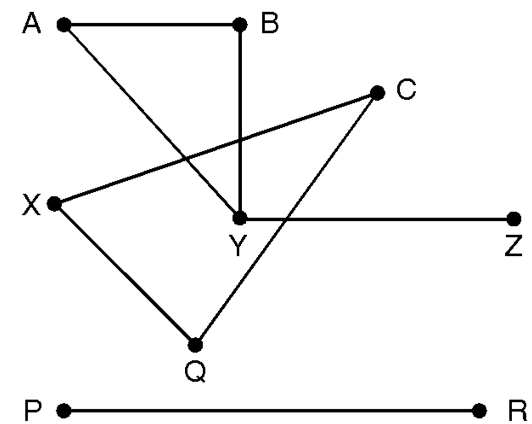


Problem

Determine whether the graph is connected or disconnected. If disconnected find its connected component.



(a)



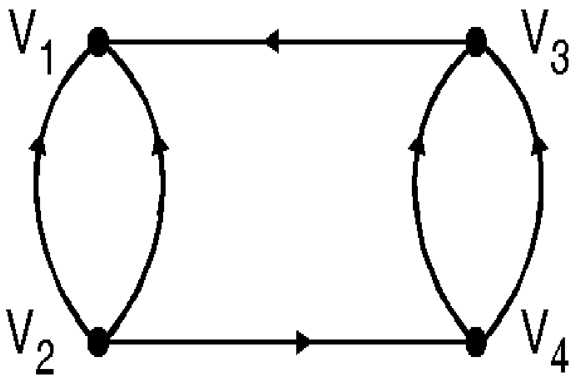
(b)

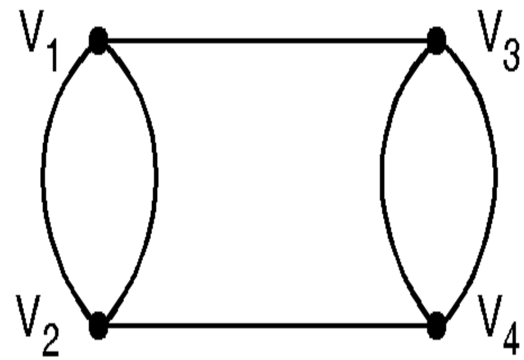
(a)Graph shown in (a) is not connected its connected components are {A, D, P, S, C} and {B, Q, R}

(b)Graph shown in (b) is not connected its connected components are {A, B, Y, Z}, {C, X, Q}, {P, R}

Multigraph

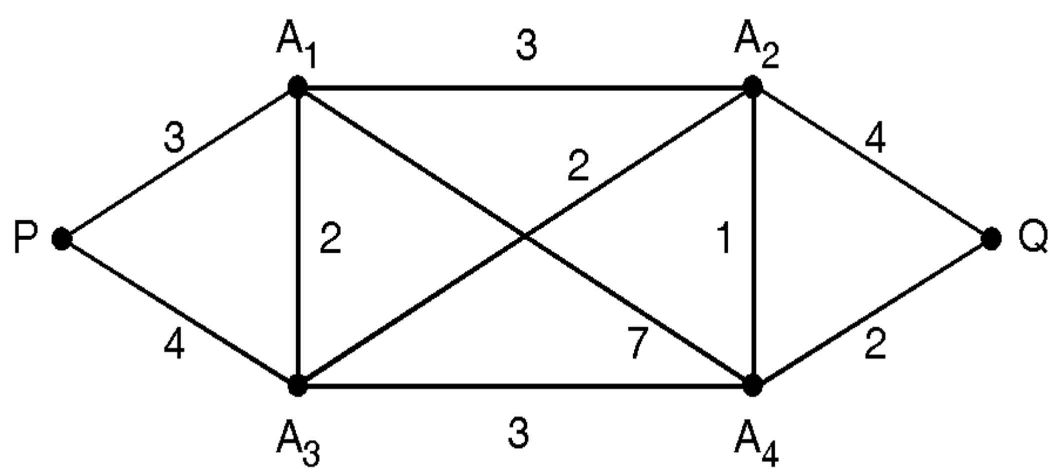
Directed graph having multiple edges between two vertices is called as **multigraph**. Undirected graph having more than one edge between two vertices is also called as **Multigraph**.





Labelled and weighted graph

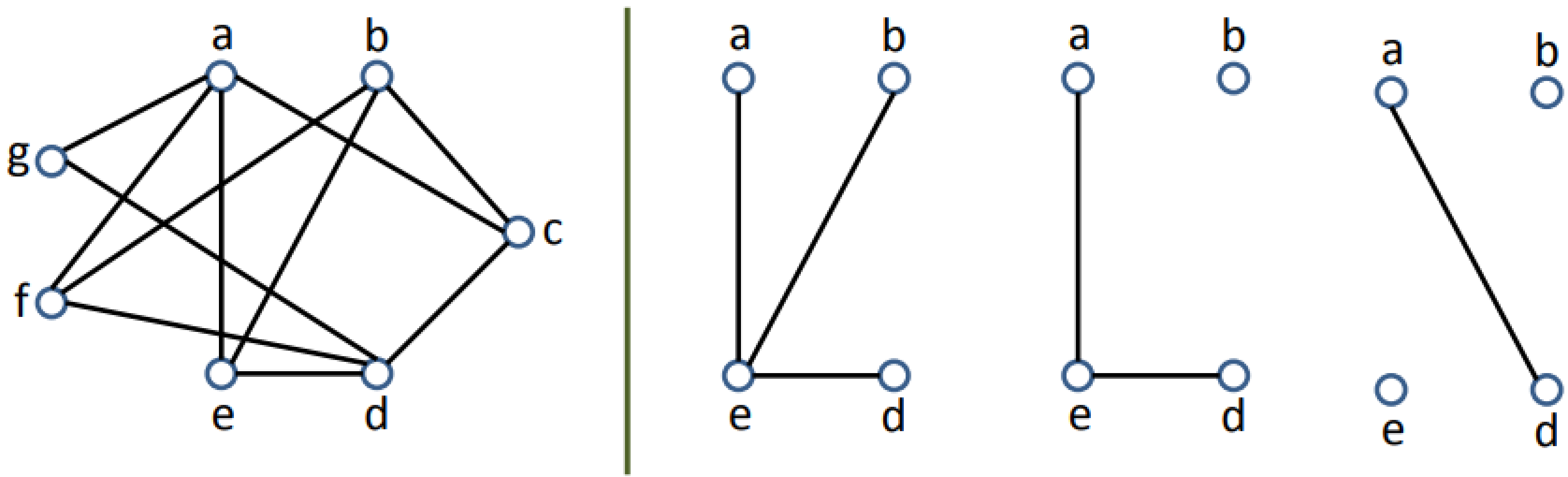
A graph G is called a **labelled graph** it its edges and /or vertices are assigned data of one kind or another. In particular, G is called a **weighted graph** if each edge 'e' of G is assigned a non–negative number called the weight or length of V.



Subgraphs

If G = (V, E) is a graph, then G' = (V', E') is called a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

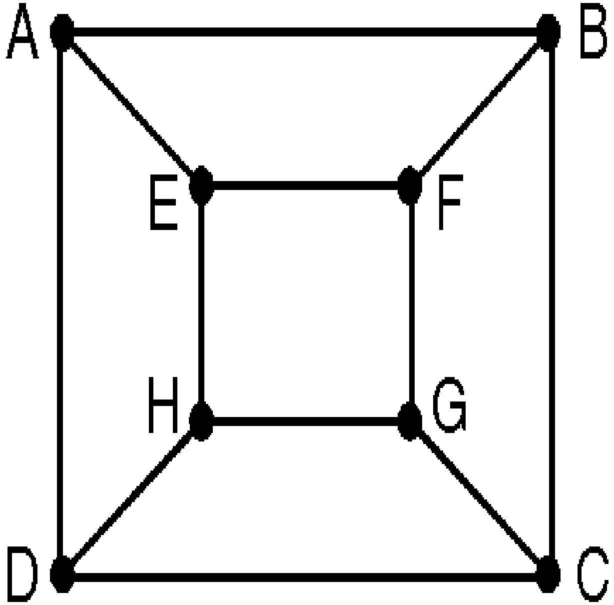
- Which one is a subgraph of the leftmost graph G ?

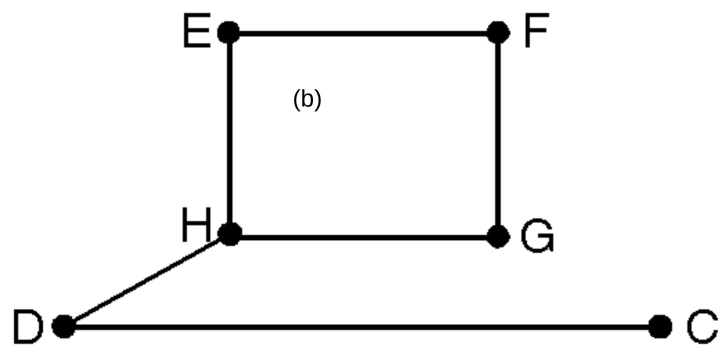


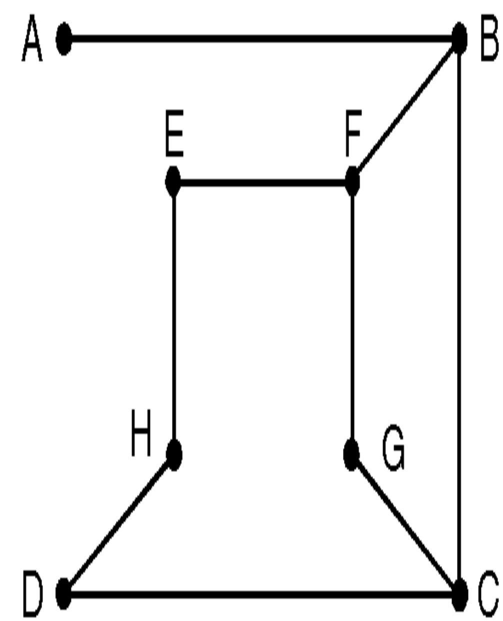
Subgraph

(a)

Let G = (V, E, γ) is a graph. Choose a subset E_1 of the edges in E and a subset V_1 of the vertices in V. So that V_1 contains all the end points of edges in E_1. Then H = (V_1, E_1, γ_1) is also a graph, where γ_1 is γ restricted to edges in E1. Such a graph H is called a **subgraph** of G.

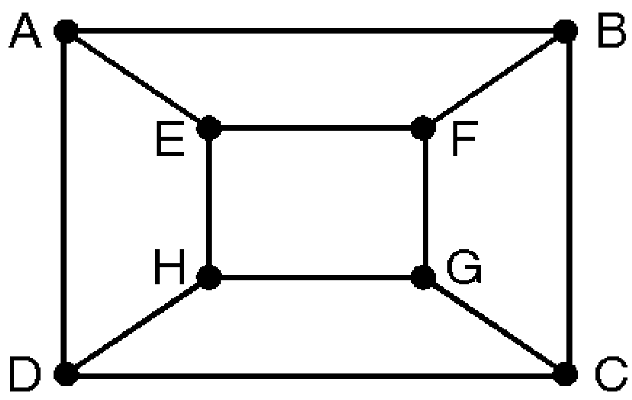


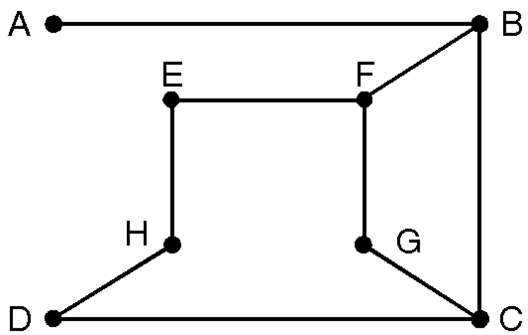




Spanning Subgraph

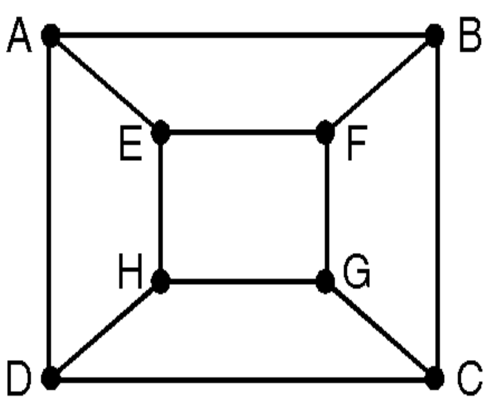
A subgraph is said to be **spanning subgraph** if it **contains all the vertices of G**.

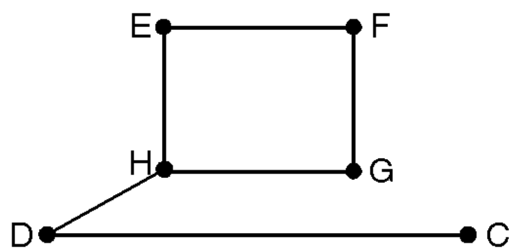


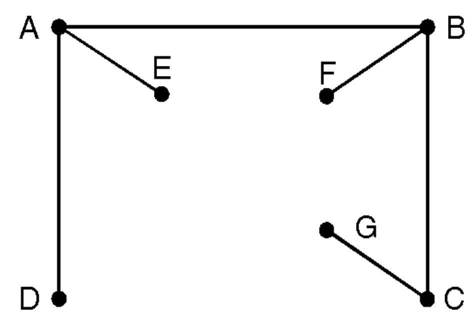


Complement of Subgraph

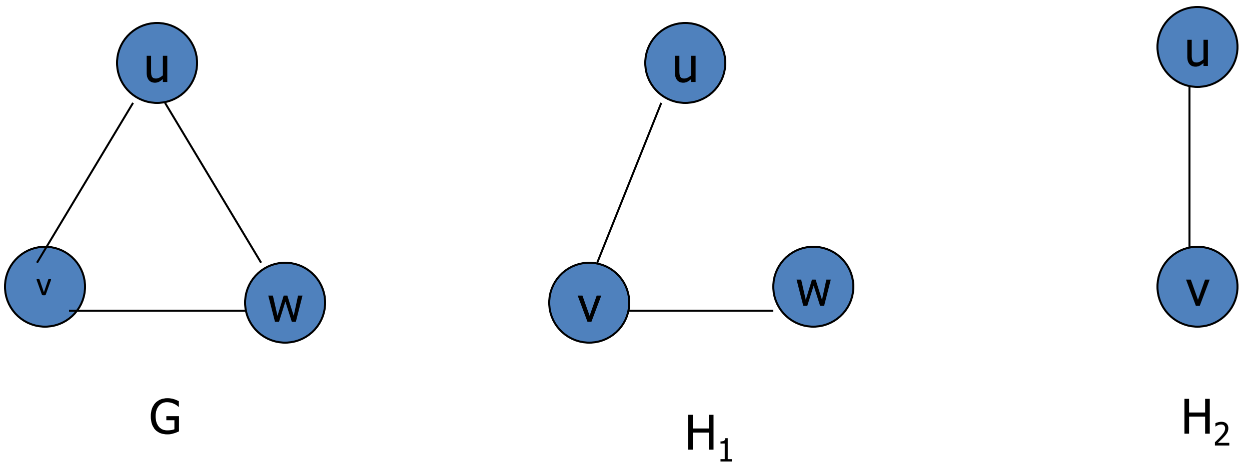
The complement of a subgraph G' = (V', E') with respect to the graph G = (V, E) is another subgraph
G" = (V", E") such that E" is equal to E – E' and V" contains only the vertices with which the edges in E" are incident.



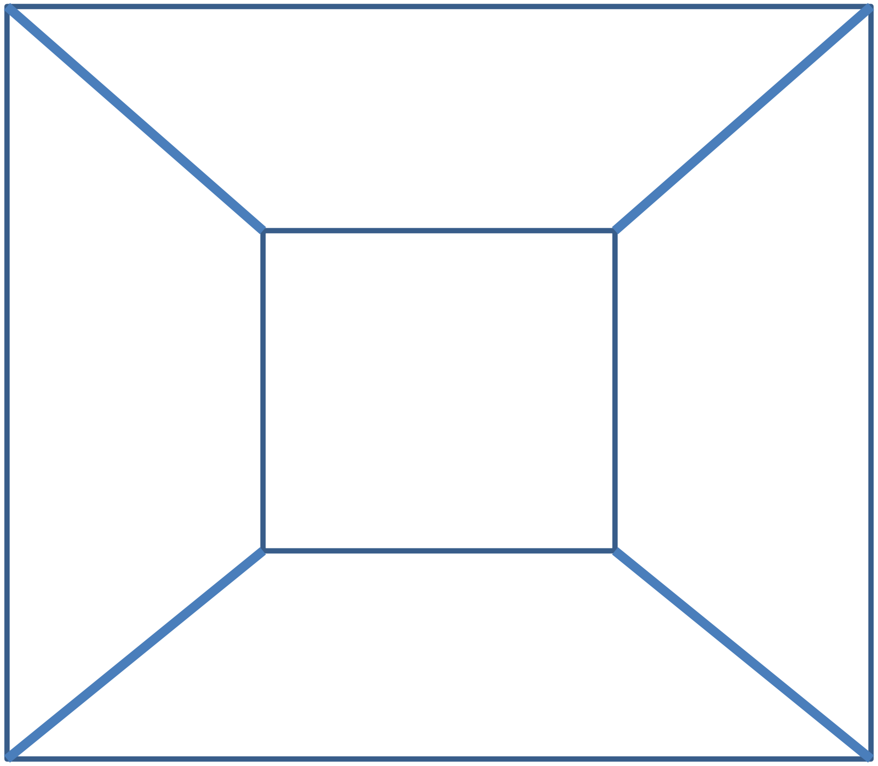


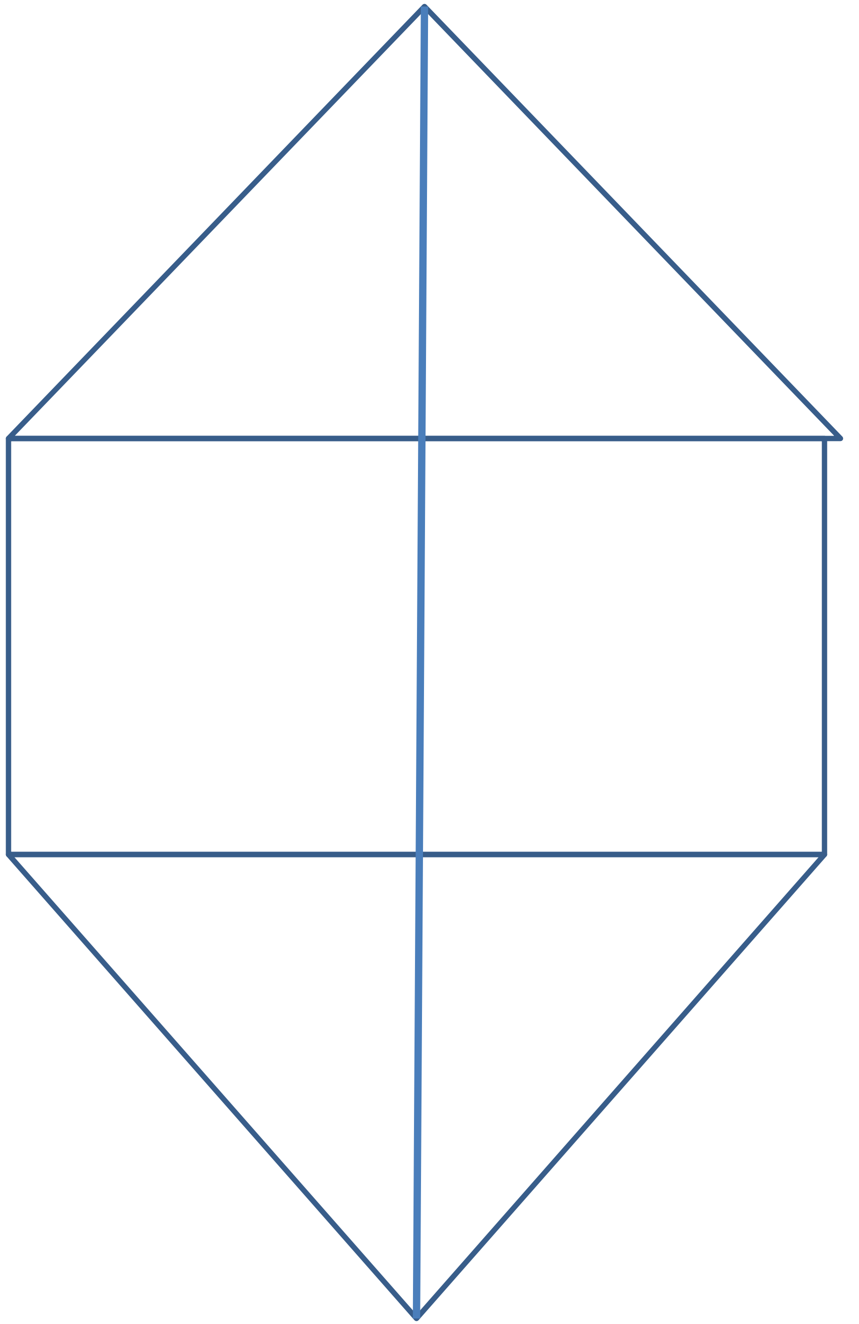


Complement of Subgraph

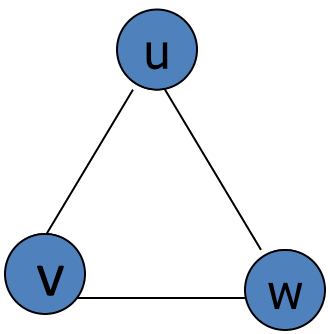


Find Sub graphs of G

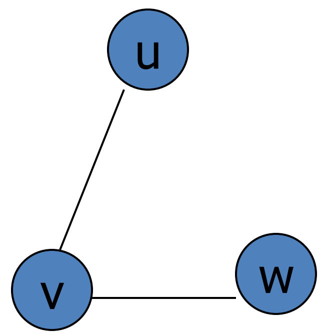




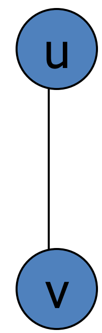
Compliment of Sub Graph



G



H_1



H_2

Handshaking Lemma

Consider a Graph G with e nos of edges and n nos of vertices , **the sum of the degrees of all vertices in G is twice the nos of edges in G**

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Problems

- Determine the number of edges in a graph with 6 nodes in which 2 of degree 4 and 4 of degree 2. Draw two such graphs
- Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining nodes have degree 4
- Is it possible to draw a simple graph with 4 vertices and 7 edges . Justify ?

- **Path** : A path is a sequence of vertices where no edge is chosen more than once
  - A path is called *simple* if no vertex repeats more than once
- **Length of Path** : Number of edges in a path is called as length of path
- **Circuit**: A circuit is a path that begins and ends with the same vertex

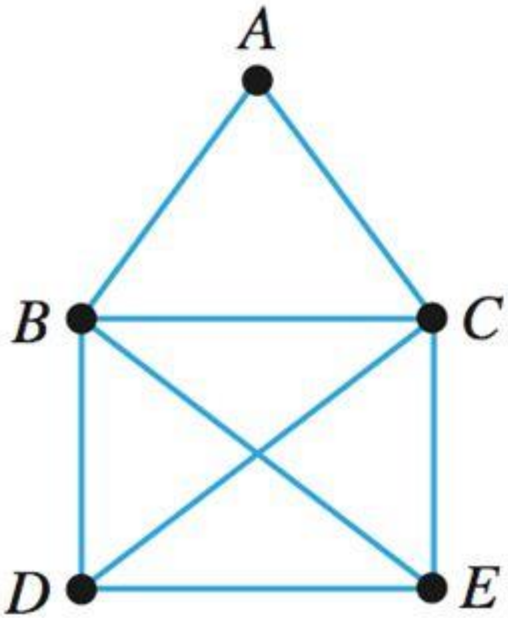
EULER PATH AND EULER CIRCUIT

- EULER PATH
  - A path in a graph G is called an Euler path if it includes every edge exactly once
- EULER CIRCUIT
  - A Euler path that is a circuit

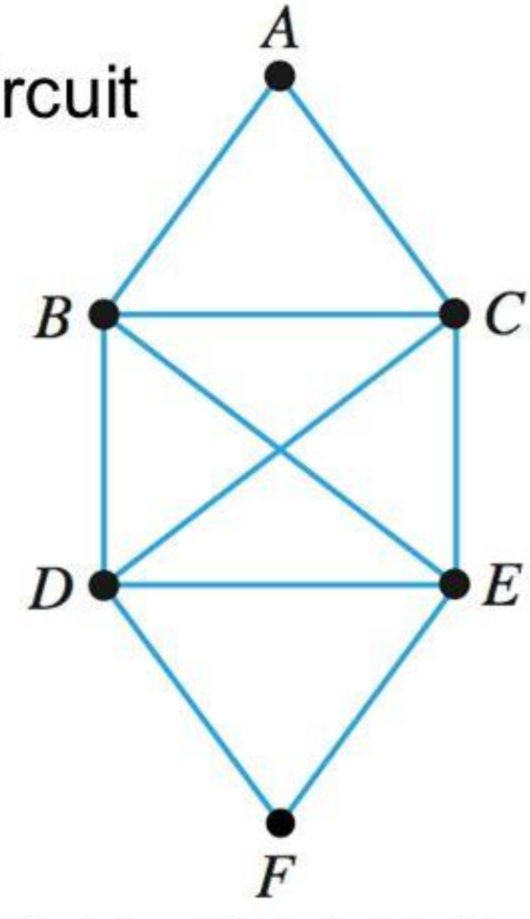
Examples

- Euler path

D, E, B, C, A, B, D, C, E



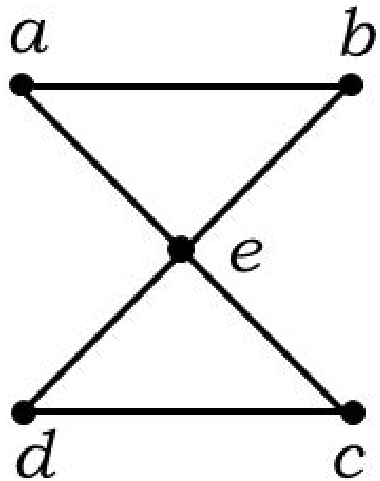
- Euler circuit

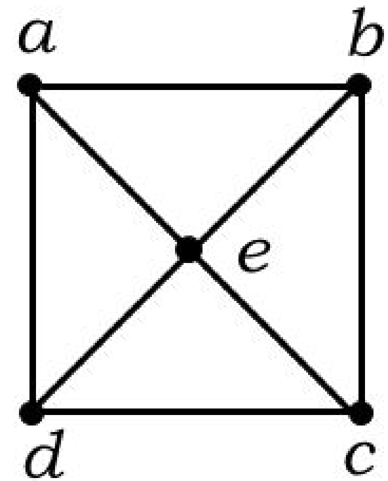


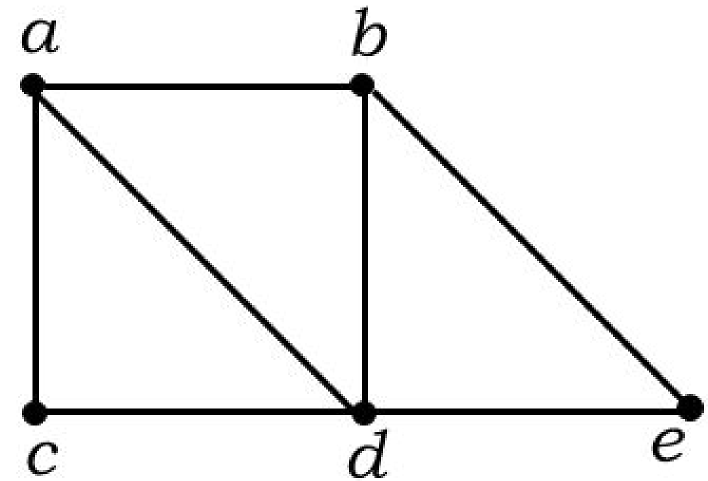
D, E, B, C, A, B, D, C, E, F, D

Example

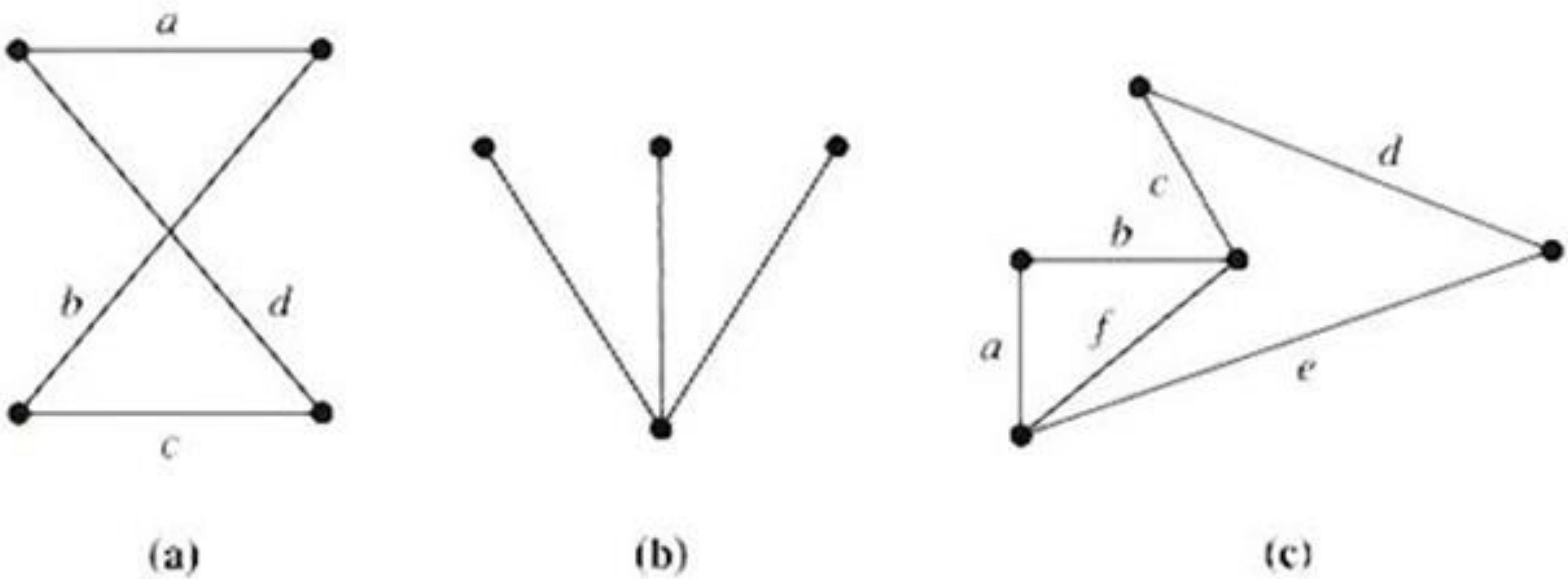
- Which of the following graphs has an Euler *circuit*?







(a, e, c, d, e, b, a)



- The path a, b, c, d in (a) is an Euler circuit since all edges are included exactly once.
- The graph (b) has neither an Euler path nor circuit.
- The graph (c) has an Euler path a, b, c, d, e, f but not an Euler circuit.

Theorem: EULER CIRCUIT

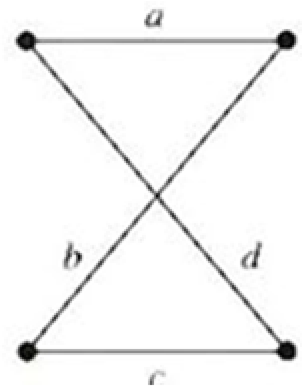
A) If graph G has a vertex of odd degree , then there can be no Euler circuit in G

B) If G is a connected graph and every vertex has an even degree then there is a Euler circuit in G

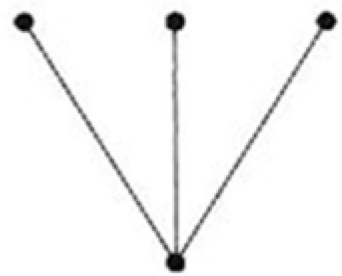
Theorem: EULER PATH

A) If a graph G has more than two vertices of odd degree then there can be no Euler path in G

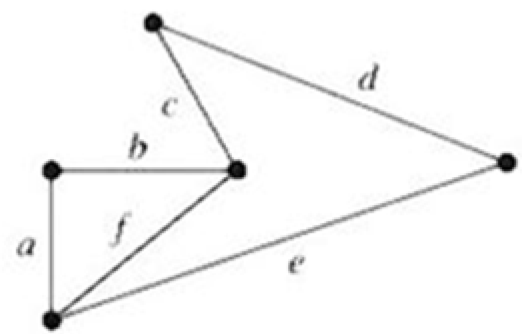
B) If G is connected and has exactly two vertices of odd degree then there is a Euler path in G



All vertices have even degree



All vertices have odd degree

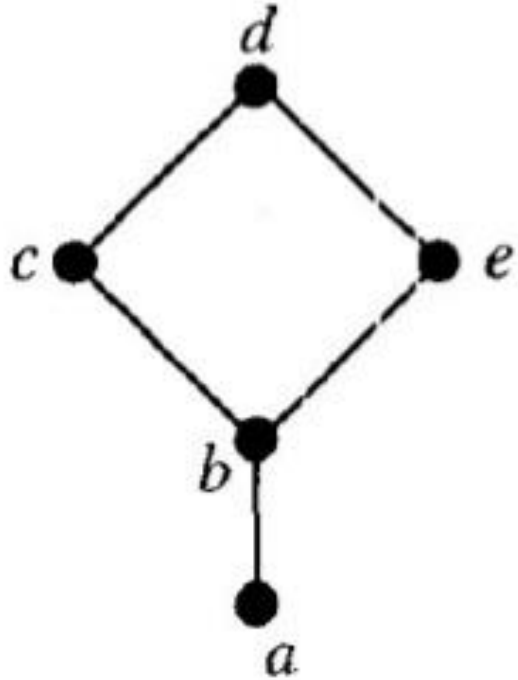


Two vertices have odd degree

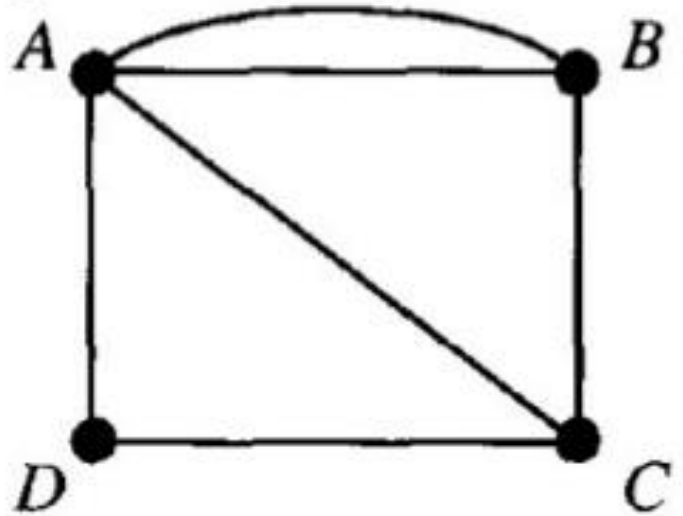
HAMILTONIAN PATH AND CIRCUIT

- A Hamiltonian path contains each **vertex exactly once**
- A Hamiltonian circuit is a circuit that contains each vertex exactly once except for the first vertex which is also the last

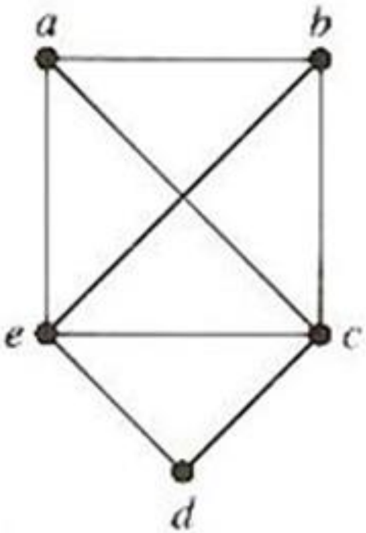
Examples



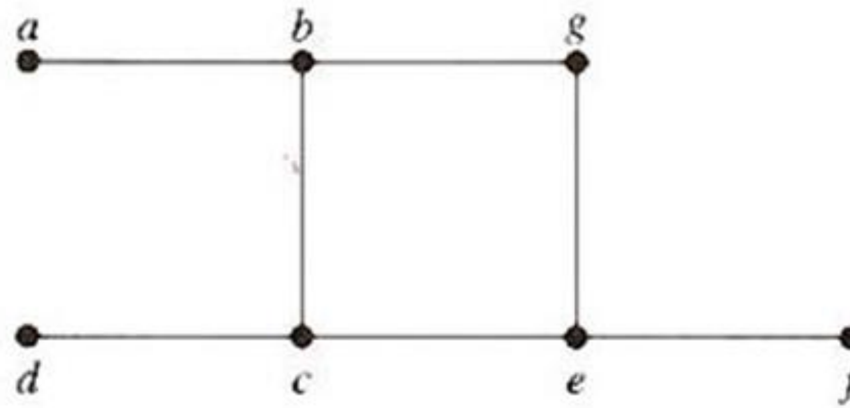
Hamiltonian path: a, b, c, d, e



Hamiltonian circuit: A, D, C, B, A


a
b
e
c
d


a
b
d
c


a
b
g
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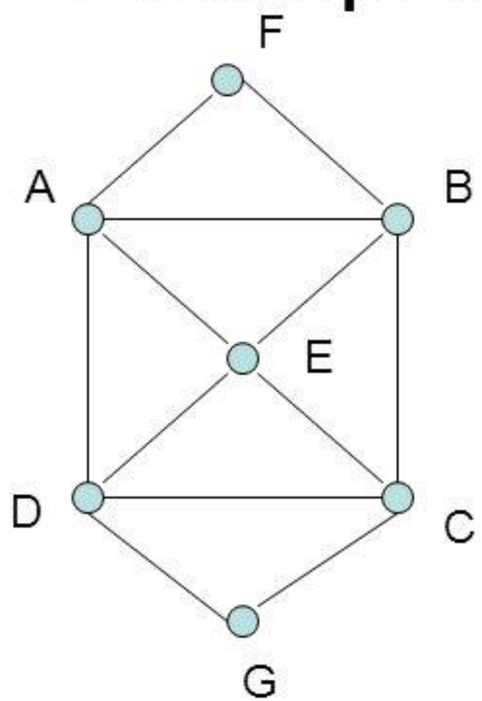
Examples of Hamilton circuits



Graph 3

Has many **Hamilton circuits**:

1) A, F, B, E, C, G, D, A
2) A, F, B, C, G, D, E, A

Has many **Hamilton paths**:

1) A, F, B, E, C, G, D
2) A, F, B, C, G, D, E

Has **Euler circuit** => Every vertex has even degree

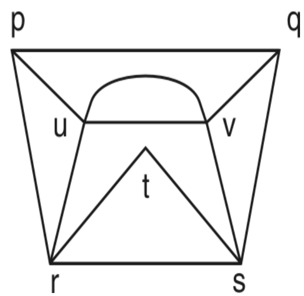
Theorem: HAMILTONIAN CIRCUIT

A) G has a Hamiltonian circuit if for any two vertices u and v of G that are not adjacent ,degree(u)+degree(v) ≥ nos of vertices

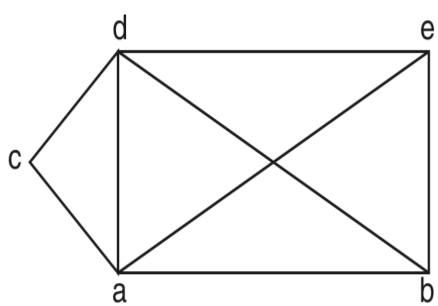
B) G has a Hamiltonian circuit if each vertex has degree greater than or equal to n/2

Problem

Determine the Eulerian and Hamiltonian path, if exists, in the following graphs.

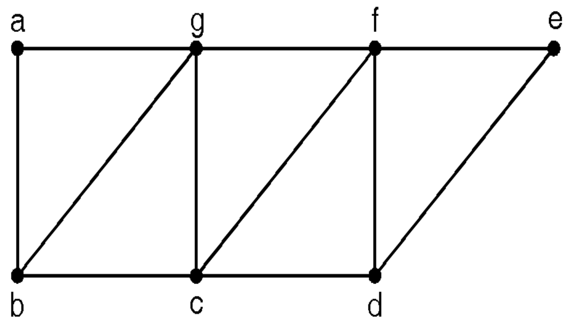


Hamiltonian path : p, u, v, q, s, t, r
Hamiltonian circuit : r, p, u, v, q, s, t, r
Eulerian path : (p, u, v, q, s, v, u, r, t, s, r, p, q)



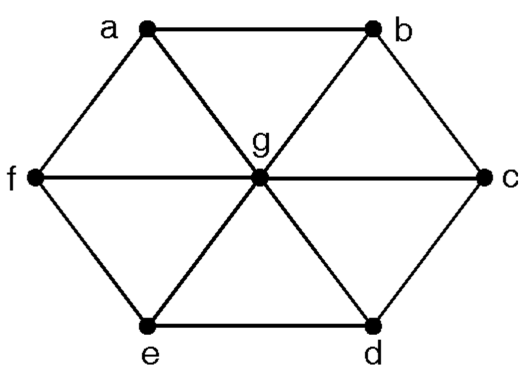
Hamiltonian path : c, d, e, b, a
Hamiltonian circuit : c, d, e , b, a, c
Eulerian path : (e, d, b, a, d, c, a, e, b)

Identify Euler path, circuit, Hamiltonian path and circuit



(a)

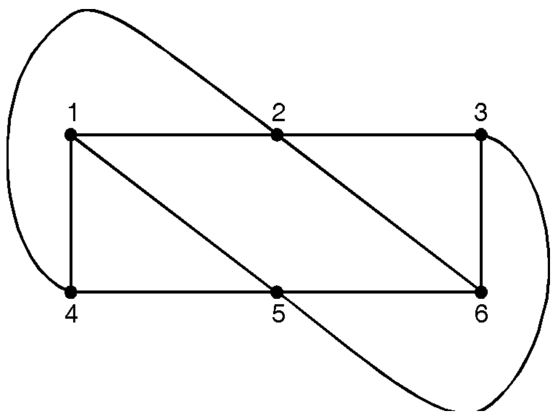
(a) two vertices b and d have odd degree.
Hence there is an Euler path.
π: b, a, g, f, e, d, c, b, g, c, f, d



(b)

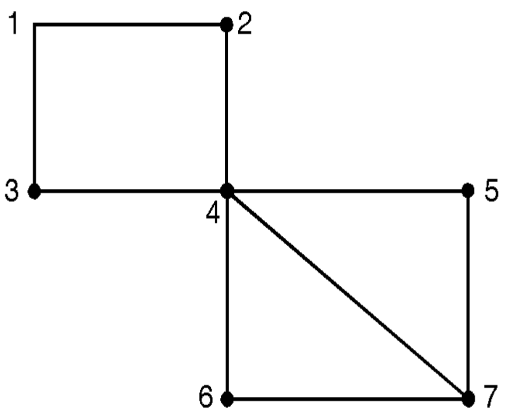
(b)6 vertices have odd degree, 3 and 1 vertex of even degree, 6.
So Euler path does not exist in this graph.

Identify Euler path, circuit, Hamiltonian path and circuit



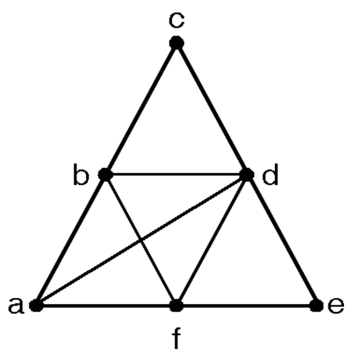
Number of vertices is 6. Each vertex has degree greater than equal to 6/2. So there is an Hamiltonian circuit.

π : 1, 4, 5, 6, 3, 2, 1

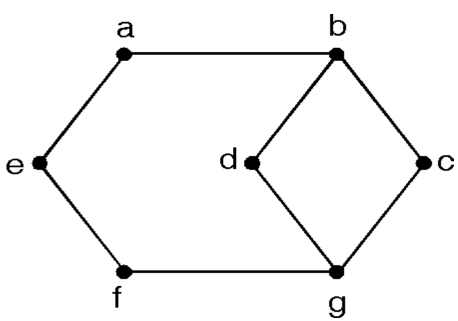


There is no Hamiltonian circuit.
But there is an Hamiltonian path
π: 3, 1, 2, 4, 6, 7, 5.

Identify Euler path, circuit, Hamiltonian path and circuit



(i) Eulerian Path : π: a, b, c, d, b, f, d, a, f, e, d
G has 2 vertices of odd degree.
Hamiltonian Circuit : a, b, c, d, e, f, a.
Hamiltonian Path : a, b, c, d, e, f



(ii) Eulerian Circuit : -
Eulerian Path : g, d, b, a, e, f, g, c, b.
Hamiltonian Path : d, b, a, e, f, g, c

Graph Isomorphism

Graphs G = (V, E) and H = (U, F) are isomorphic if we can set up a bijection f : V → U such that

x and y are adjacent in G

⇔ f(x) and f(y) are adjacent in H

- Function f is called isomorphism
  - Same nos of vertices
  - Same nos of edges
  - Equal nos of vertices with a given degree
  - Adjacency of vertices

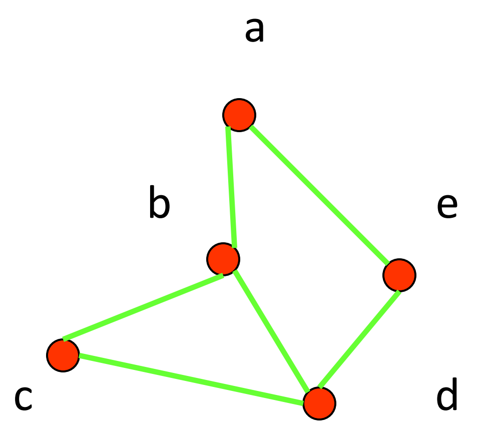
Graph - Isomorphism

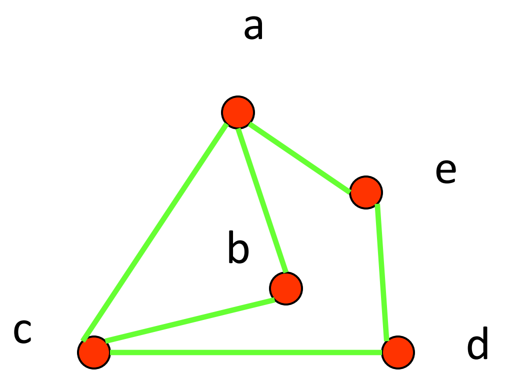
Representation example: G1 = (V1, E1) , G2 = (V2, E2)

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$,

Isomorphism of Graphs

- **Example I:** Are the following two graphs isomorphic?

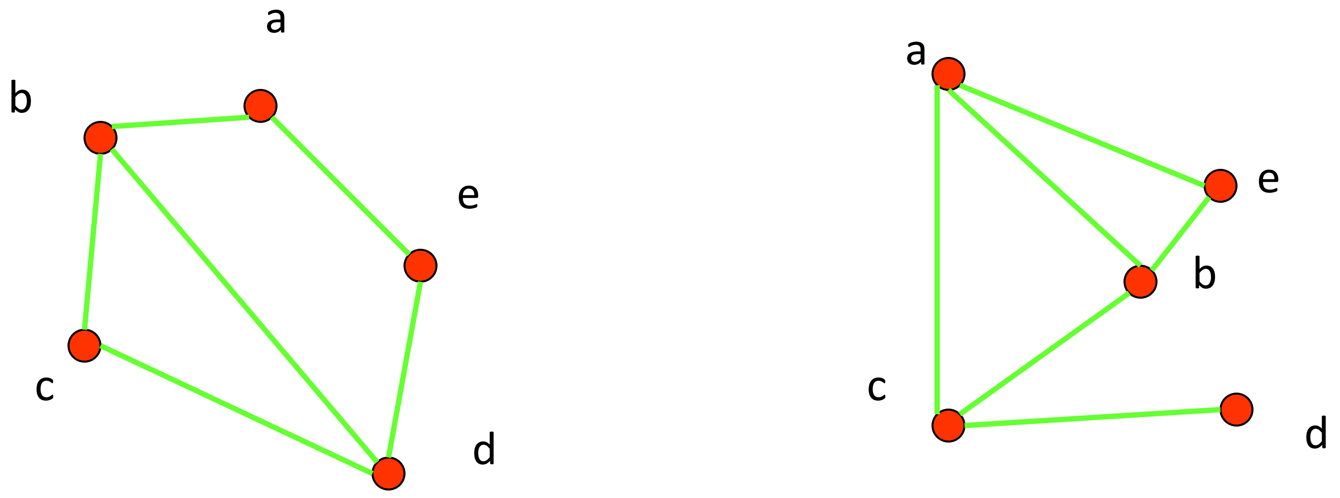




Solution: Yes, they are isomorphic, because they can be arranged to look identical. You can see this if in the right graph you move vertex b to the left of the edge {a, c}. Then the isomorphism f from the left to the right graph is: f(a) = e, f(b) = a, f(c) = b, f(d) = c, f(e) = d.

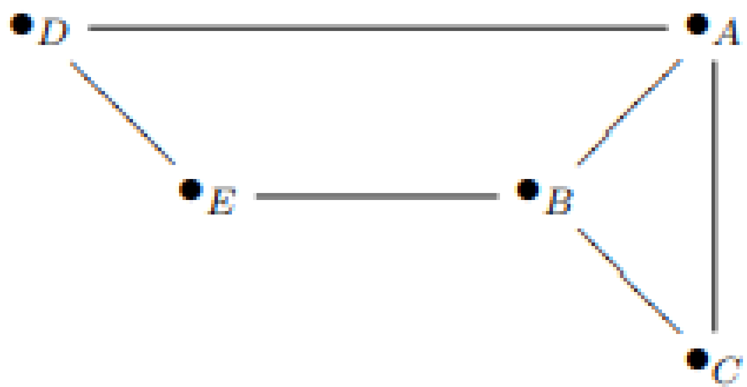
Isomorphism of Graphs

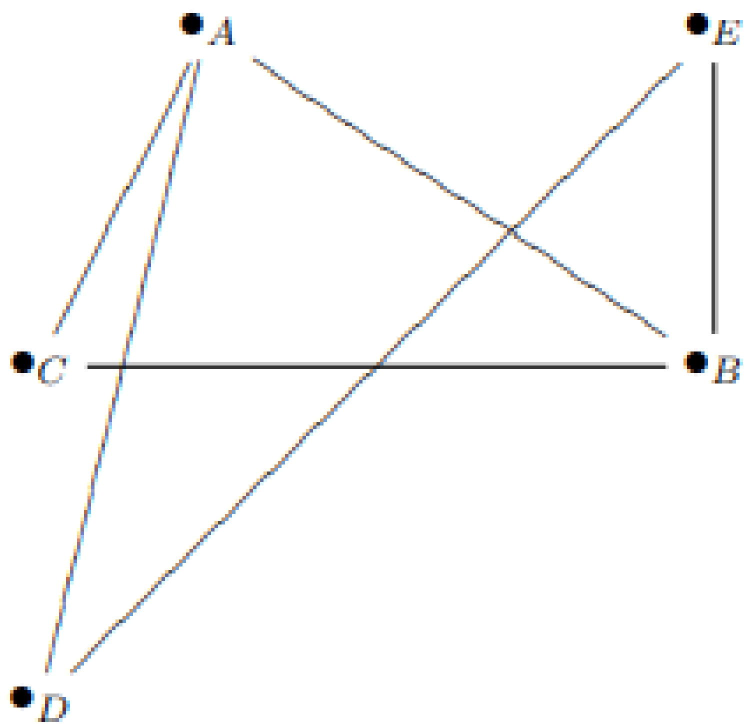
- **Example II:** How about these two graphs?



Solution: No, they are not isomorphic, because they differ in the degrees of their vertices.

Vertex d in right graph is of degree one, but there is no such vertex in the left graph.





A is adjacent to: B, C, D

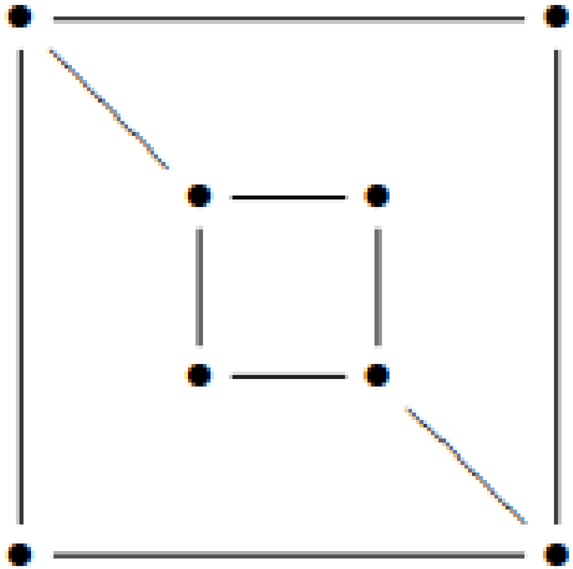
B is adjacent to: A, C, E

C is adjacent to: A, B

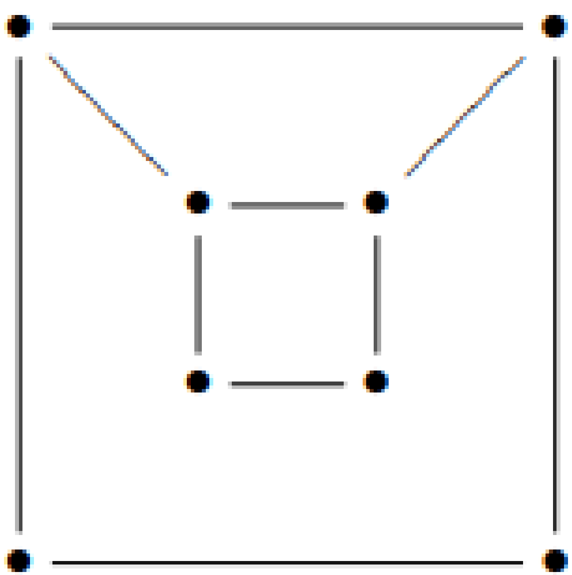
D is adjacent to: A, E

E is adjacent to: B, D

G :



H :



Both graphs contain

8 vertices and 10 edges

Nos of vertices of degree 2 = 4

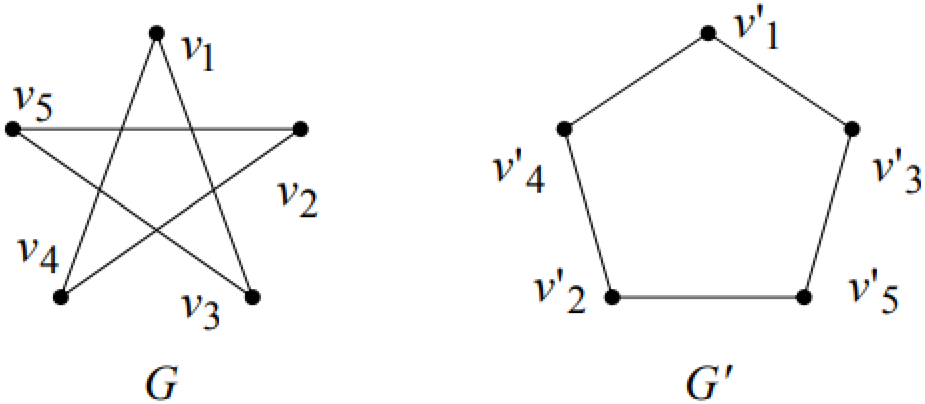
Nos of vertices of degree 3 = 4

Adjacency : There exists **no** vertex of degree 3 whose adjacent vertices have same degree in both graphs

So its not ISOMORPHIC

Isomorphism of Graphs

Example IV: Are the following two graphs isomorphic?

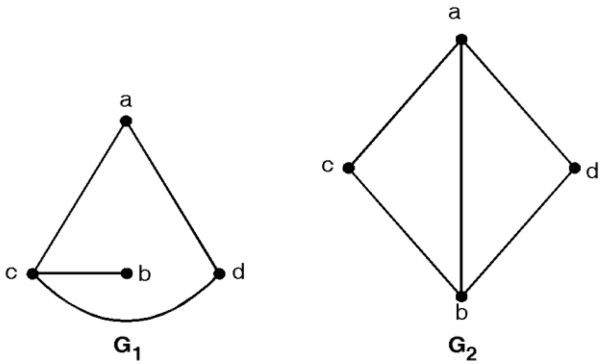


Solution: Both graphs have 5 vertices and 5 edges. All vertices have degree 2.

$f : V \to V'$	
V	V'
v_1	v_1'
v_2	v_2'
v_3	v_3'
v_4	v_4'
v_5	v_5'

Isomorphism of Graphs

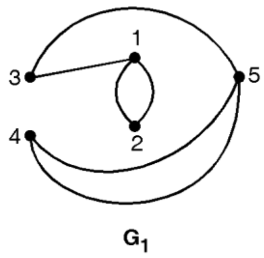
Example V: Are the following two graphs isomorphic?

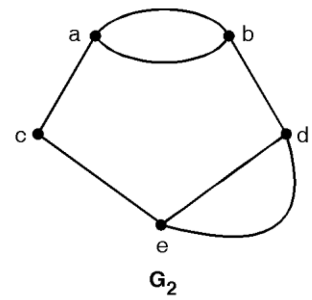


Solution: Here G1 and G2 both have 4 vertices but G1 has 4 edges and G2 has 5 edges. Hence G1 is not isomorphic to G2.

Isomorphism of Graphs

Example VI: Are the following two graphs isomorphic?





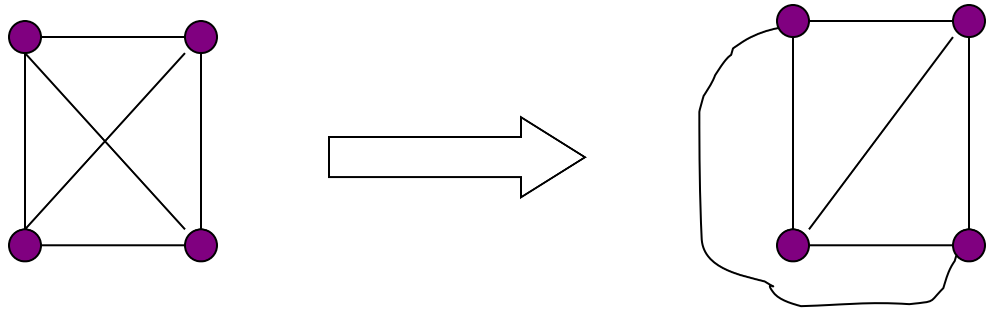
Solution: G1 and G2 both have 5 vertices but G1 has 6 edges while G2 has 7 edges. Hence G1 $\ncong$ G2. That is G1 is not isomorphic to G2.

Planar Graphs

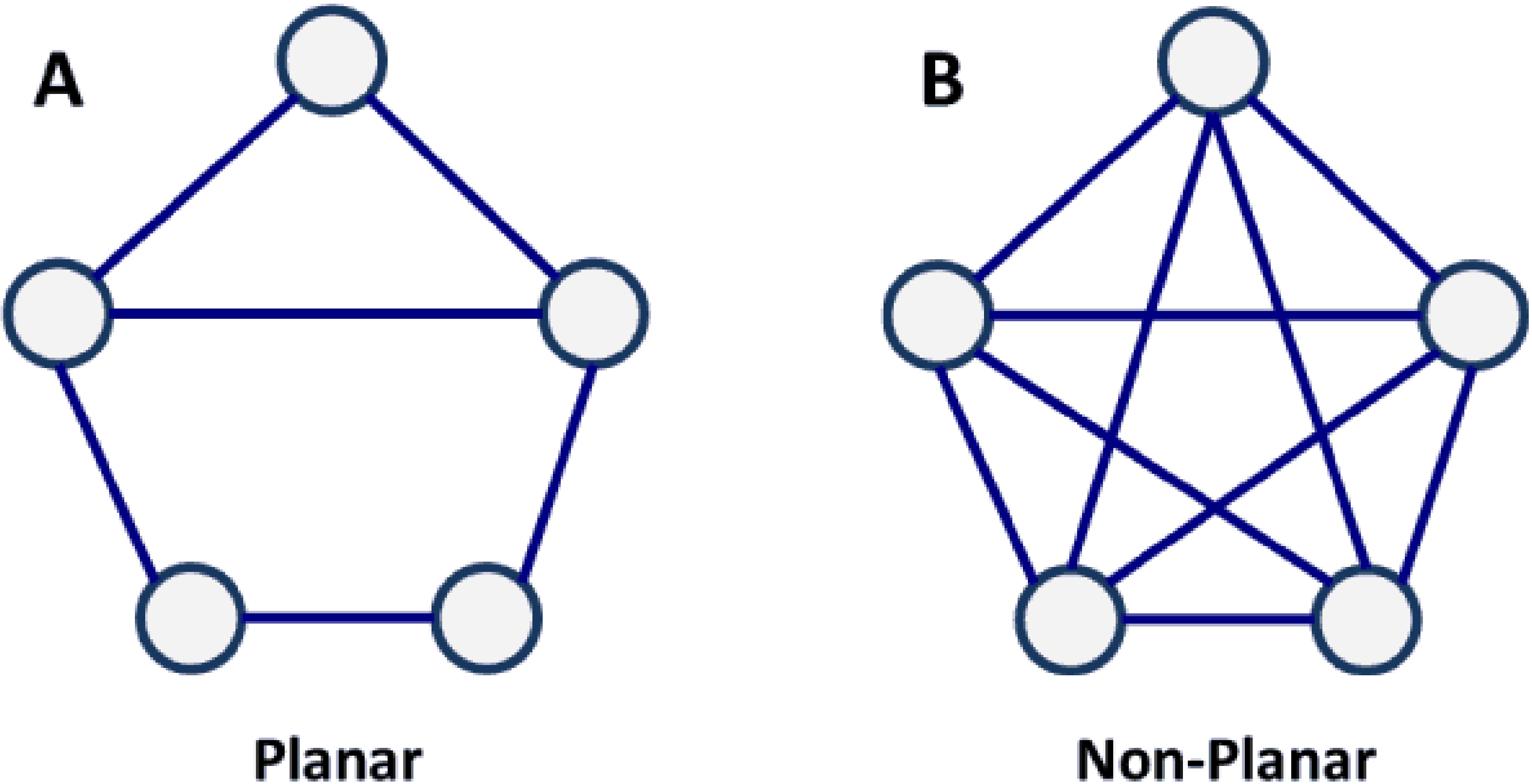
- A graph (or multigraph) *G* is called *planar* if *G* can be drawn in the plane with its edges intersecting only at vertices of *G*, such a drawing of *G* is called an *embedding* of *G* in the plane.

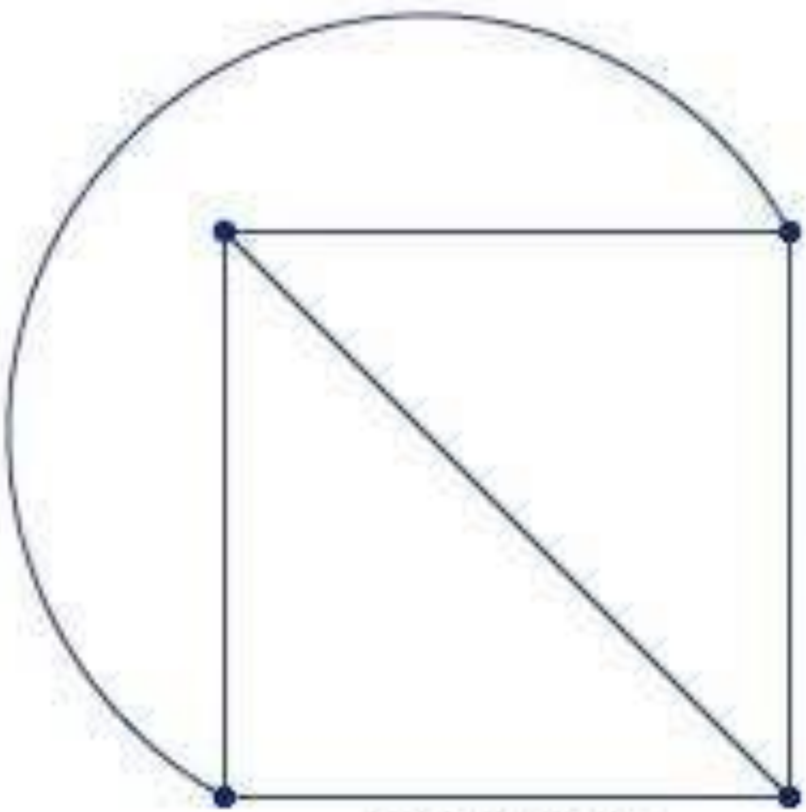
Application Example: VLSI design (overlapping edges requires extra layers), Circuit design (cannot overlap wires on board)

Representation examples: $K1$,$K2$,$K3$,$K4$ are planar, Kn for $n>4$ are non-planar

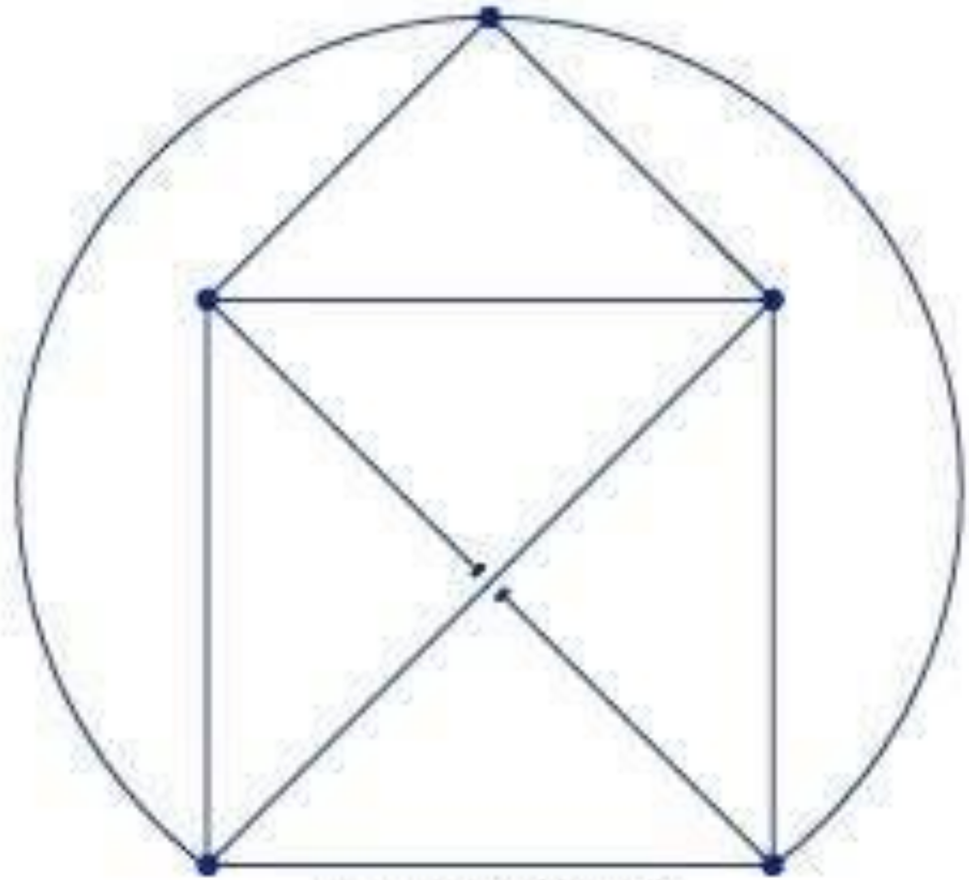


K_4


A
Planar
B
Non-Planar



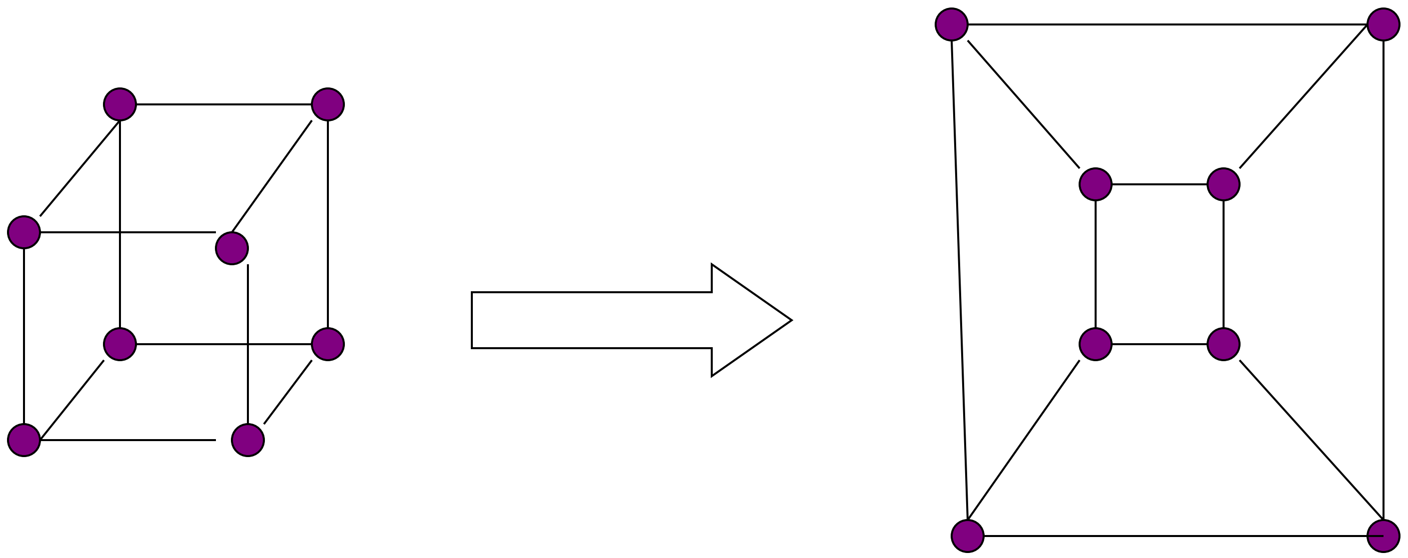
K_4 is planar



K_5 is not planar

Planar Graphs

- Representation examples: Q_3



Planar Graphs

Theorem : *Euler's planar graph theorem*

For a **connected** planar graph or multigraph:

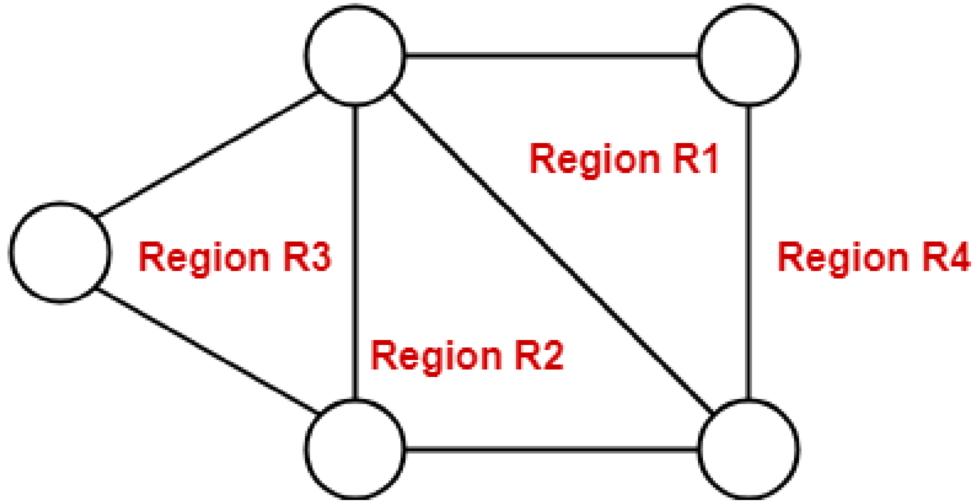
$$v - e + r = 2$$

number of vertices

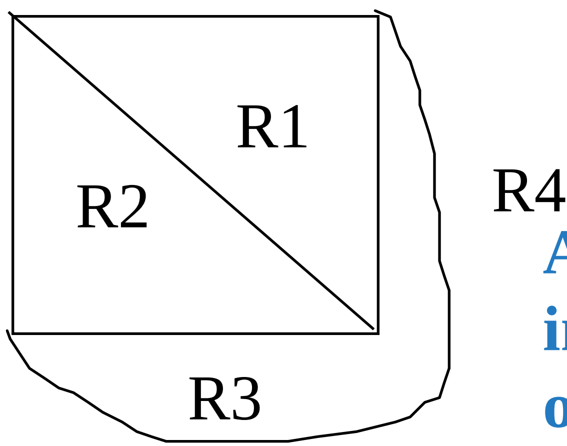
number of edges

number of regions

Planar Graphs



K_4

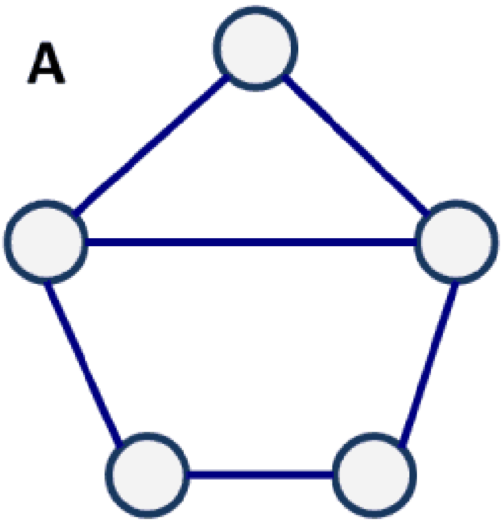


A planar graph divides the plane into several regions (faces), one of them is the infinite region.

$v=4, e=6, r=4, v-e+r=2$

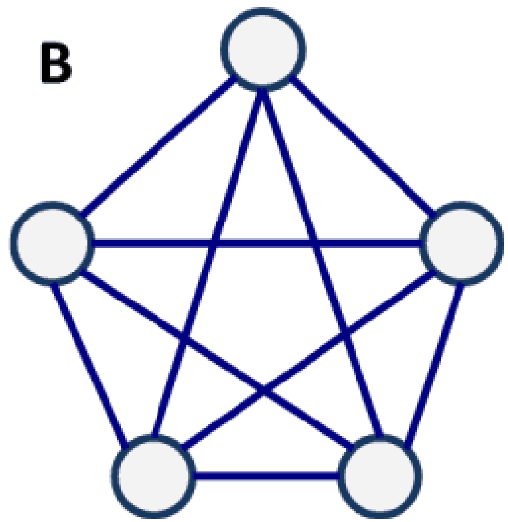
Examples

A



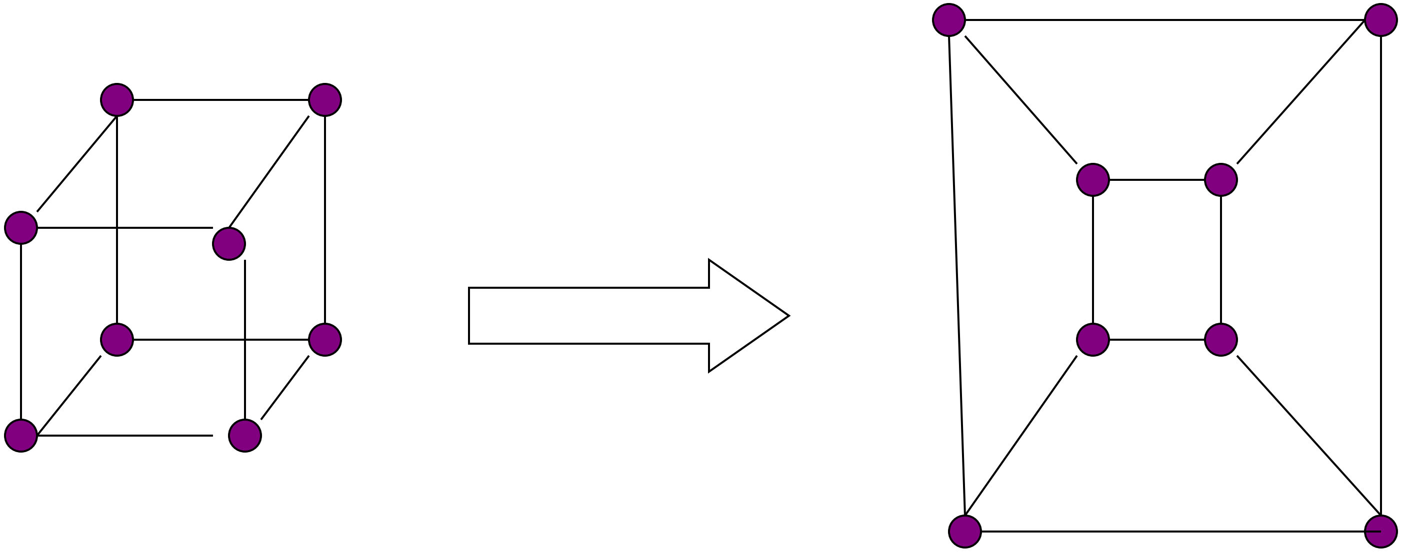
Planar

B

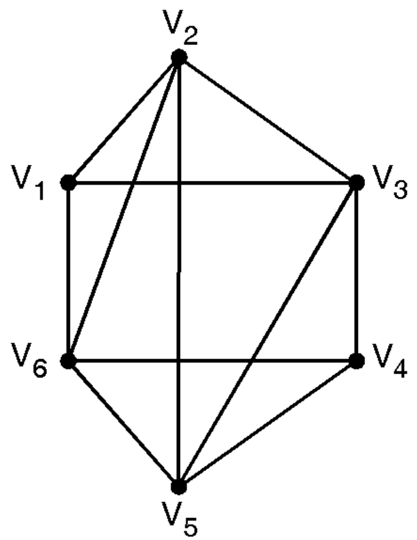


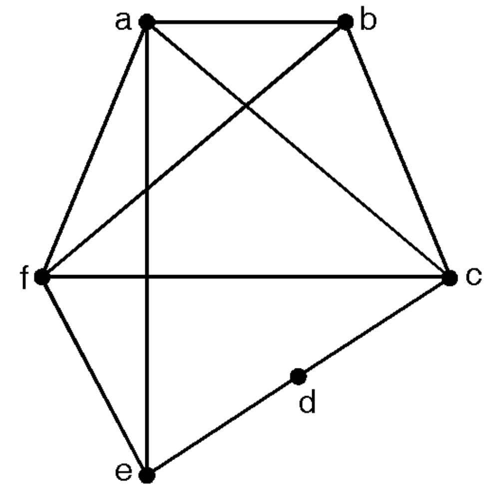
Non-Planar

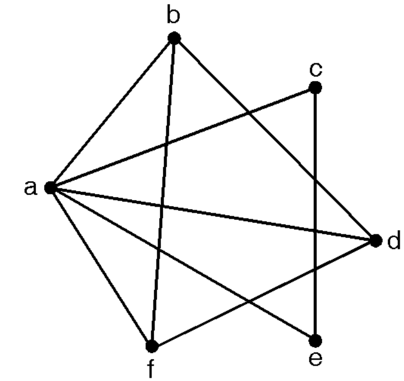
Planar Graphs Example

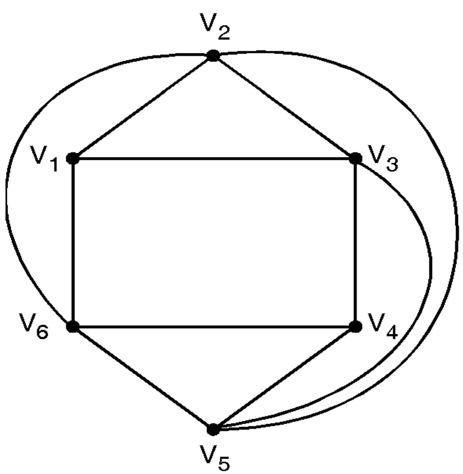


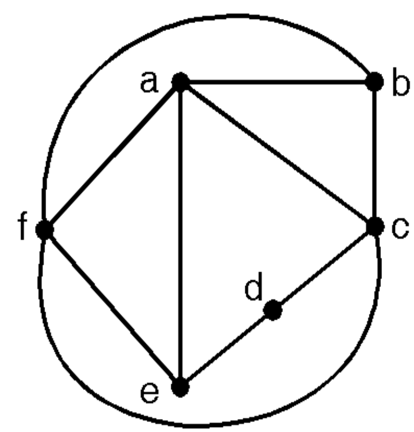
Q. 1) By drawing the graph, show that following graphs are planar graphs

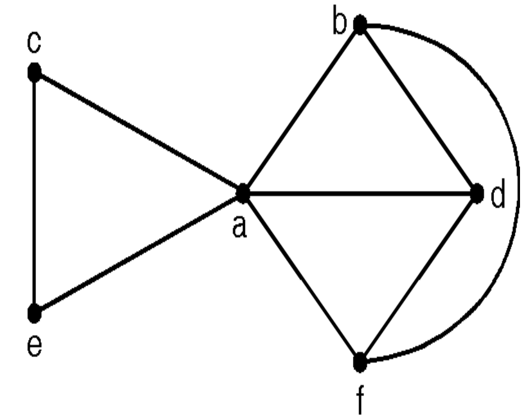












Q. 2 : How many edges must a planar graph have if it has 7 regions and 5 nodes. Draw one such graph.

Soln. :

According to Euler's formula, in a planar graph

$$v - e + r = 2$$

where v, e, r are the number of vertices, edges and regions in a planar graph.

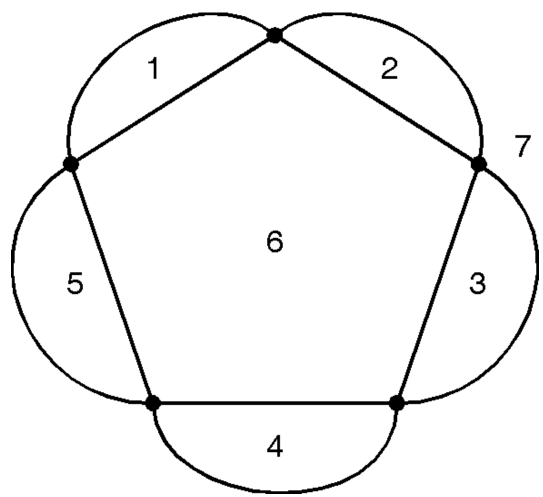
Here $v = 5, r = 7, e = ?$

$$v - e + r = 2$$

$$5 - e + 7 = 2$$

$$e = 10$$

Hence the given graph must have 10 edges.



Q. 3 : Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a multi-graph.

Soln. :

Given v = 6, e = 10

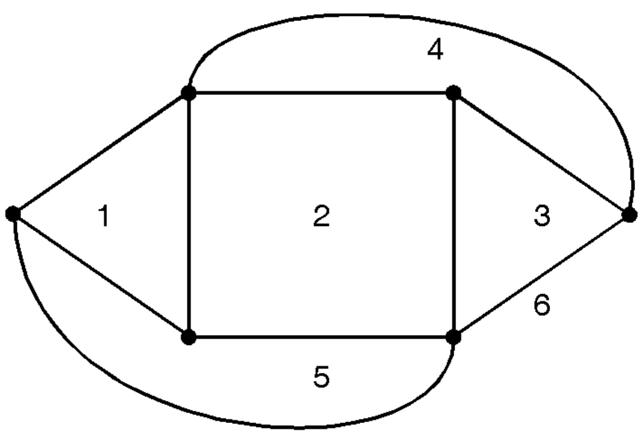
Hence by Euler's formula for a planar graph

$$v - e + r = 2$$

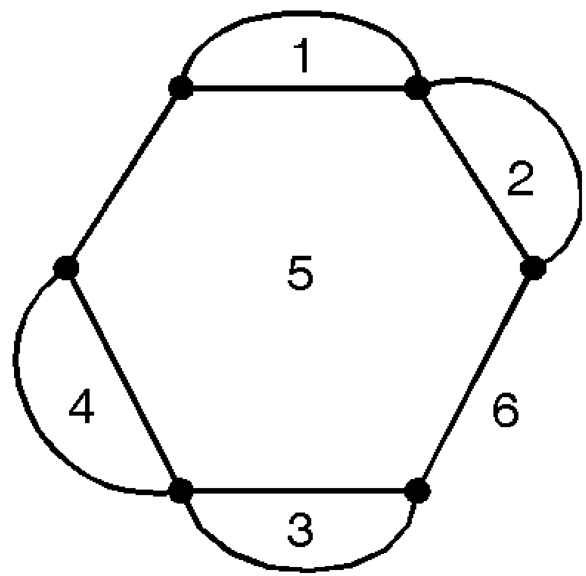
$$6 - 10 + r = 2$$

$$r = 6$$

Hence the graph should have 6 regions.



(a) Simple Graph



(a) Multi-Graph

Q. 4 : A connected planar graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ?

Soln. :

By handshaking lemma

$$\Sigma d(v_i) = 2e$$

where $d(v_i)$ = degree of ith vertex

e = number of edges

For given graph

$$2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 2.e$$

$$28 = 2e$$

$$e = 14$$

\ There are 14 edges.

Ex. 5 : Suppose that a connected planer graph has 20 vertices, each of degree 3 into how many regions does a representation of this plan graph split the plane ?

Soln. :

$$|V| = 20 = \text{number of vertices}$$

$$\text{degree of each vertex} = 3$$

By hand shaking Lemma

$$\Sigma\, d(V_i) = 2e$$

$$20 \times 3 = 2e$$

$$\Rightarrow \quad e = 30$$

By Euler's theorem,

$$|V| - |E| + |R| = 2$$

$$20 - 30 + |R| = 2$$

$$|R| = 12$$

Planar graph will split the plane in 12 regions.