MODULE 4
POSETS AND LATTICE

Syllabus

4.1 Partial ordered relations (Posets) ,Hasse diagram

4.2 Lattice, sub-lattice

4.3 Types of Lattice ,Boolean Algebra

POSETS(Partially Ordered Sets)

Partially ordered relation : A relation R on a set A is called **partial order** if R is reflexive, anti-symmetric and transitive.

Poset : The set A together with the partial order R is called a **partially ordered set** or simply a **poset.** It is denoted by (A, R).

Examples

1. Let A be a set of positive integers and let R be a binary relation such that (a, b) is in R *if a divides b.*

Since any integer divides itself. R is reflexive. Since a divides b means b does not divide a unless a = b, R is an anti-symmetric relation. Since a divides b and b divides c then a divides c, so R is transitive relation. Consequently, R is a partial ordered relation.

2. Let A be a collection of subsets of set S. The relation '$\subseteq$' of set inclusion is a partial order on A. So (A, $\subseteq$) is a poset because set has desired properties

(i) $A \subseteq A$ for any set A

(ii) if $A \subseteq B$ and $B \subseteq A$ then $A = B$

(iii) if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Dual of Poset

Let R be a partial order on a set A, and let R^{-1} be the inverse relation of R. Then R^{-1} is also a partial order. The poset (A, R^{-1}) is called the **dual of the poset** (A, R) and the partial order R^{-1} is called the dual of the partial order R.

Product Partial Order

If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset with partial order $\leq$ defined by $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.

The partial order $\leq$ defined on the Cartesian product $A \times B$ is called the product partial order.

A={1,2,3} B={0,4} R: <=

R={(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)} (A,R)=poset

R={(0,0),(0,4),(4,4)} (B,R)=poset

$A \times B$={(1,0),(1,4),(2,0),(2,4),(3,0),(3,4)}

R={ ((1,0),(1,4)), ((1,0),(2,0)), ((1,0),(2,4)),…………..}

$(A \times B$,R)=poset

R=product partial order.

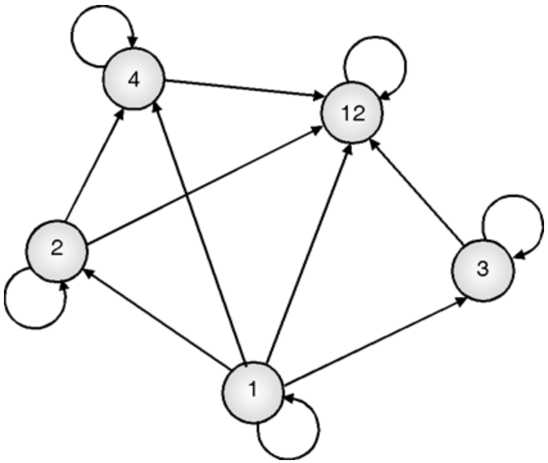
Hasse Diagram

Procedure for drawing Hasse Diagram :

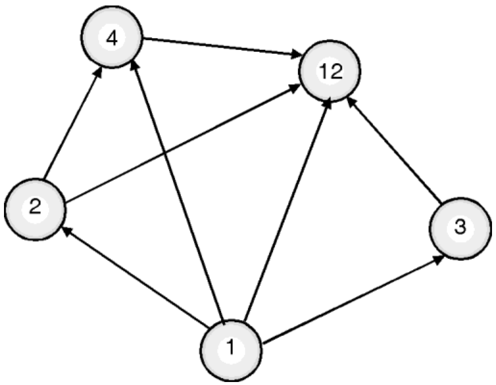
1. Draw the digraph of given relation
2. Delete all cycles from digraph
3. Eliminate all edges that are implied by the transitive relation. For ex. If a R b, b R c then a R c so eliminate (a, c) edge.
4. Draw the digraph of a partial order with all edges pointing upward so that arrows may be omitted from edges.
5. Finally replace the circles representing the vertices by dots.

Such a graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing upward is known as the “Hasse digraph” of the relation.

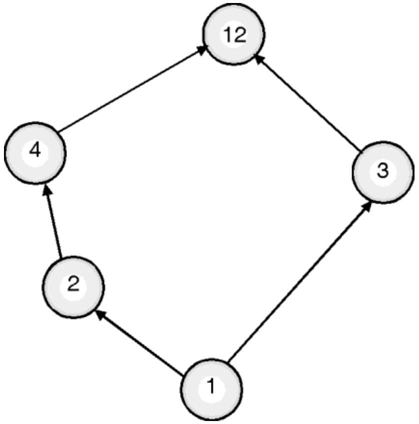
Draw Hasse diagram for the following relations on set A={1, 2, 3, 4, 12}
R={(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (4, 12), (1, 3), (1, 4), (1, 12), (2, 4), (2, 12), (3,12)}



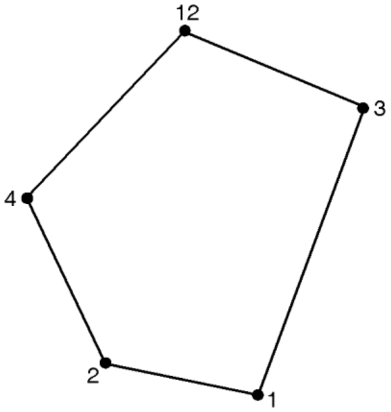
Step 1: Remove cycles



Step 2: Remove transition edge. Eliminated transitive edges (1, 4), (2, 12), (1, 12).



Step 3:
Circles are replaced by dots. Arrows are also removed.



Let A = {a, b, c, d} and R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i)Prove that R is partial order.

(ii)Draw Hasse diagram of R.

Solution:

(i) R = {(a, a), (a, c), (a, d), (b, b), (b, c), (b, d), (c, d), (c, d), (d, d)} R is reflexive because it contain (a, a), (b, b), (c, c), (d, d).

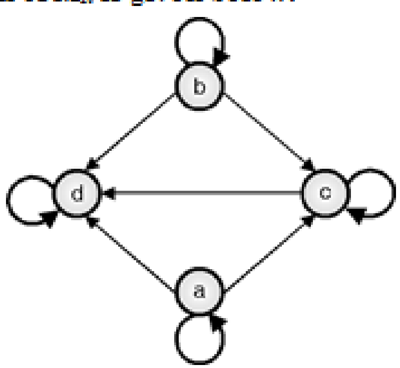
R is antisymmetric because it contain a and b such that if $a \neq b$, then $a \not R\, b$ or $b \not R\, a$.

R satisfies this condition, hence R is anti-symmetric.

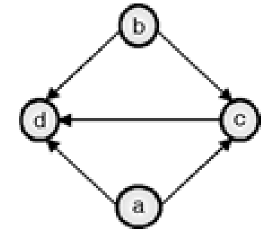
R is transitive since it contain (a, d) and (b, d).

$\therefore$ R is partial order.

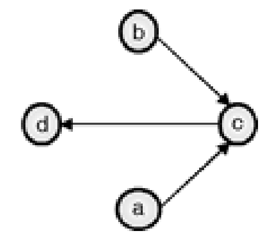
(ii) Diagraph of M_R is given below.



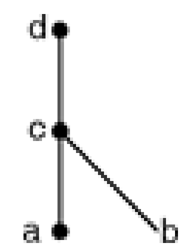
Step 1 : Remove reflexive relation



Step 2 : Remove transitive relation i.e. (a, d) and (b, d).



Step 3 : Circles are replaced by dots and all edges are pointing upward. Arrows are removed.



More Examples

Draw a Hasse diagram for A, R= (divisibility relation) where

(i) A = {1, 2, 3, 4, 5, 6, 7, 8}

R={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(2,4),(2,6),(2,8),(3,6),(4,8),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8)}

(ii) A = {1, 2, 3, 5, 11, 13}

(iii) A = {2, 3, 4, 5, 6, 30, 60}

(iv) A = {1, 2, 3, 6, 12, 24}

(v) A = {1, 2, 4, 8, 16, 32, 64}

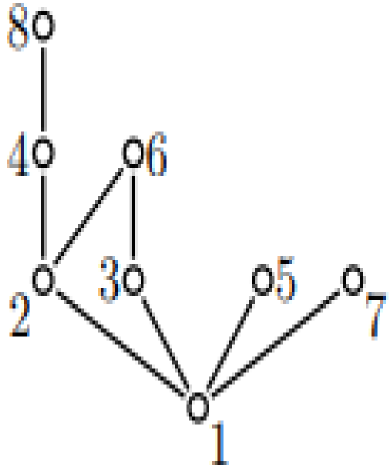
(vi) A = {2, 4, 6, 12, 24, 36}

- A={1,2,3,6,12,24}, R is divisibility relation

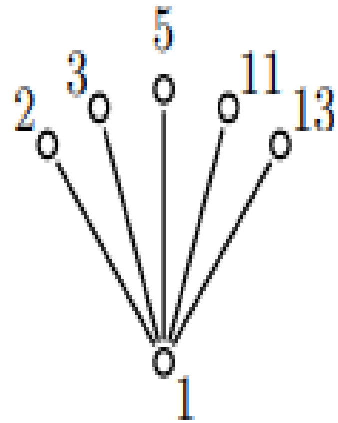
R={(1,1),(1,2),(1,3),(1,6),(1,12),(1,24),(2,2),(2,6),(2,12),(2,24),(3,3),(3,6),(3,12),(3,24),(6,6),(6,12),(6,24),(12,12),(12,24),(24,24)}

Solution:

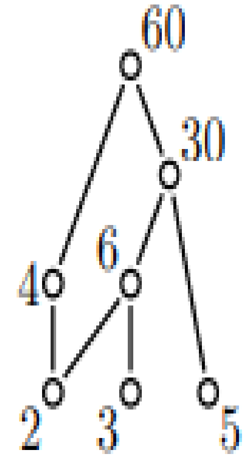
(i):



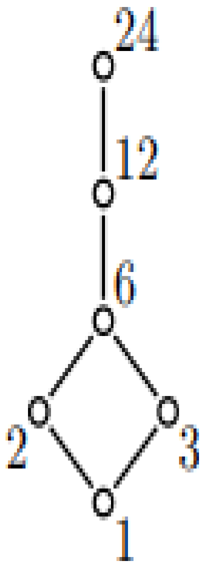
(ii):



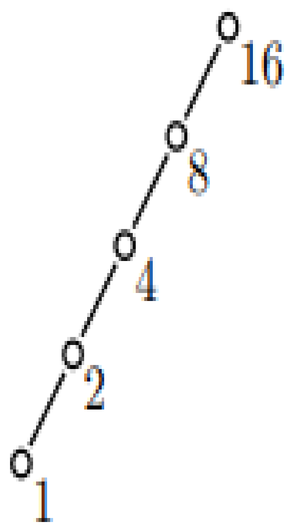
(iii):



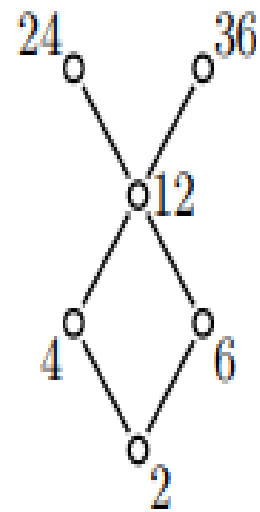
(iv):



(v):



(vi):



Chain and Antichain

Let (A, ≤) be a partially ordered set. A subset of A is called a **chain** if every two elements in the subset are related. The number of elements in a chain is called the **length** of the chain.

A subset of A is called an 'antichain' if no two distinct elements in the subset are related.

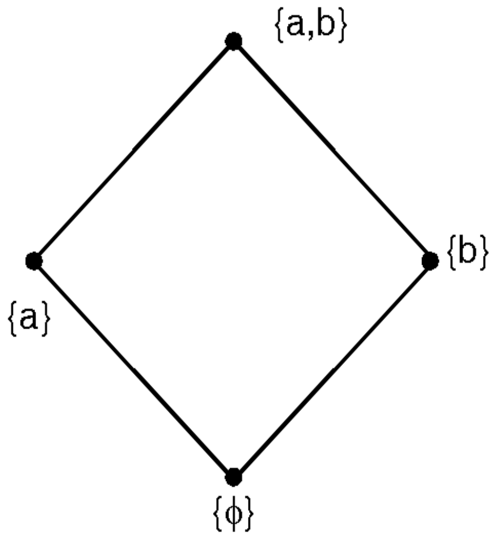
A partially ordered, set (A, ≤) is called a "totally ordered set" if A is a chain. In this case, the binary relation ' ≤ ' is called a **total ordering relation**.

Example

Let A = {a, b} and consider its poset (P(a), $\subseteq$)
Then
P(a)={{null set},{a},{b},{a,b}}

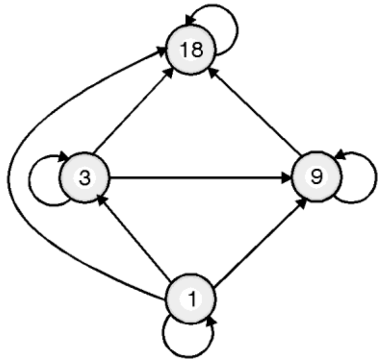
{{ϕ}, {a}, {a, b}}, {{ϕ}, {b}, {a, b}}, {{ϕ}, {a}}, {{ϕ}, {b}}, {{a}, {a, b}}, {{b}, {a, b}} are chains

and {{a}, {b}} is anti-chain.

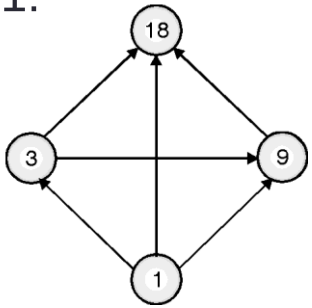


Ex. Draw the Hasse diagram of the following sets under partial ordering relation "divides" and indicate those which are chains.
(a) {1, 3, 9,18}

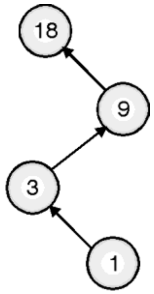
Solution: Partial ordered relation "divides" for the given set {1, 3, 9, 18} is
R={(1, 1), (1, 3), (1, 9), (1, 18), (3, 3), (3, 9), (3, 18), (9, 9), (9, 18), (18, 18)}
Digraph for the above relation is as shown



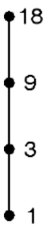
Step 1:



Step 2:



Step 3:



A is a chain
Poser (A, “divides”) is called a "totally ordered set" In this case, the binary relation ‘divides' is called a **total ordering relation**.

Maximal Element, Minimal Element

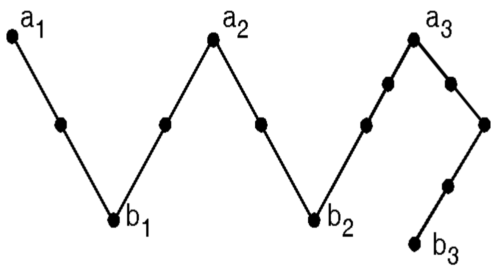
An element $a \in A$ is called a ‘**maximal element’** of A if there is no element c in A such that $a < c$.

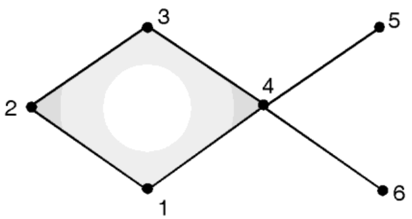
An element $b \in A$ is called a ‘**minimal element**' of A if there is no element c in A such that $c < b$.

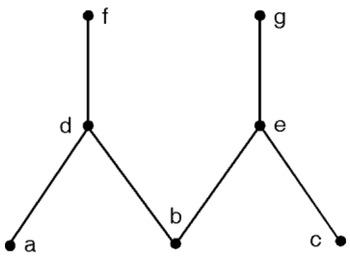
It follows immediately that if $(A, \leq)$ is a poset and $(A, \geq)$ is its **dual poset.**

An element $a \in A$ is a maximal element of $(A, \geq)$ if and only if a is a minimal element of $(A, \leq)$ Also 'a' is a minimal element of $(A, \geq)$ if and only if it is a maximal element of $(A, \leq)$.

Examples







Maximal elements : a1, a2, a3

Minimal elements : b1, b2, b3

Maximal elements : 3, 5

Minimal elements : 1, 6

Maximal elements : f, g

Minimal elements : a, b, c

Greatest Element, Least Element

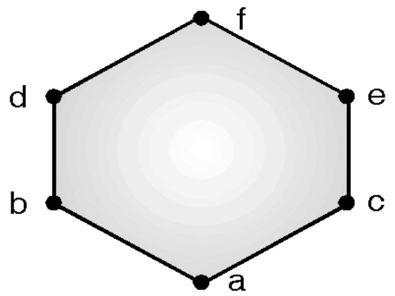
An element $a \in A$ is called a **greatest element** of A if $x \leq a$ for all $x \in A$. An element $a \in A$ is called a **least element** of A if $a \leq x$ for all $x \in A$

An element a of $(A, \leq)$ is a greatest (or least) element if and only it is a least (or greatest) element of $(A, \geq)$.

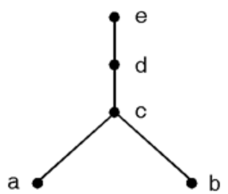
A poset has atmost one greatest element and atmost one least element.

The greatest elements of a poset if it exists, is denoted by **I** and is often called the **unit element**. similarly, the least element of a poset, if it exists, is denoted by '**O** ' and is often called the **zero element**.

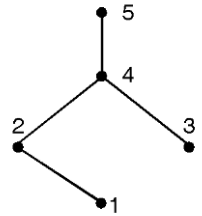
Examples



Greatest element **I** = f
Least element **O** = a



Greatest element **I** = e
Least element **O** = none



Greatest element **I** = 5
Least element **O** = none

Example

Consider the poset {3, 5, 9, 15, 24, 45}, for divisibility relation.

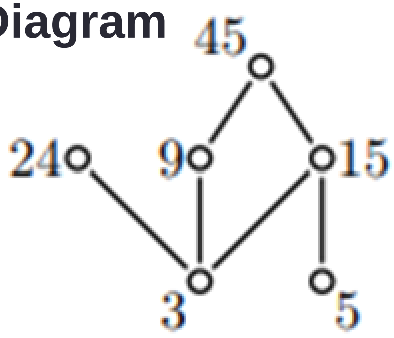
(i) Draw its Hasse diagram.

(ii) Find its maximal, minimal, greatest and least elements when they exist.

(iii) Find maximal, minimal, greatest and least elements of the set M = {3, 9, 15}, when they exist.

Solution:

i) Hasse Diagram



ii) Maximal Elements: 24,45
Minimal Elements: 3,5
Greatest, Least elements do not exist

iii) Maximal Elements: 9,15
Minimal Elements: 3
Greatest element DNE
Least element : 3

Upper Bound, Lower Bound

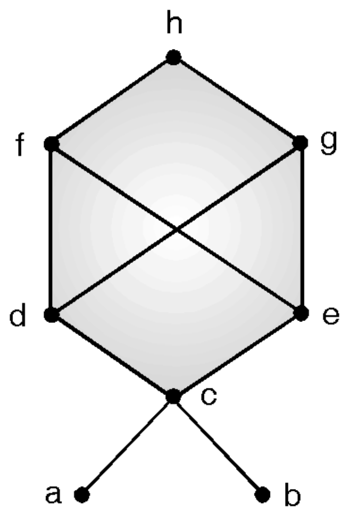
Upper Bound : Consider a poset A and a subset B of A. An element a $\in$ A is called an **upper bound** of B if b $\leq$ a for all b $\in$ B.

Lower Bound : An element a Î A is called a **lower bound** of B if a $\leq$ b for all b$\in$ B.

Least Upper Bound (LUB) : Let A be a poset and B be a subset of A. An element a $\in$ A is called a **least upper bound (LUB)** of B if a is an upper bound of B and a $\leq$ a', whenever a' is an upper bound of B. Thus a = (LUB) (B) if b $\leq$ a for all b $\in$ B and if whenever a' $\in$ A is also an upper bound of B. Then a $\leq$ a'.

Greatest Lower Bound (GLB) : Similarly, an element a $\in$ A is called a **greatest lower bound (GLB)** of B if a is a lower bound of B and a' $\leq$ a. whenever a' is a lower bound of B. Thus a = GLB

Example



Ex. : A = {a, b, c, d, e, f, g, h}

B1 $\subseteq$ A, B2 $\subseteq$ A

(i) B1 = {a, b}

(ii) B_2 = {c, d, e}

Solution:

(i) Upper bounds of set B1 are c, d, e, f, g and h

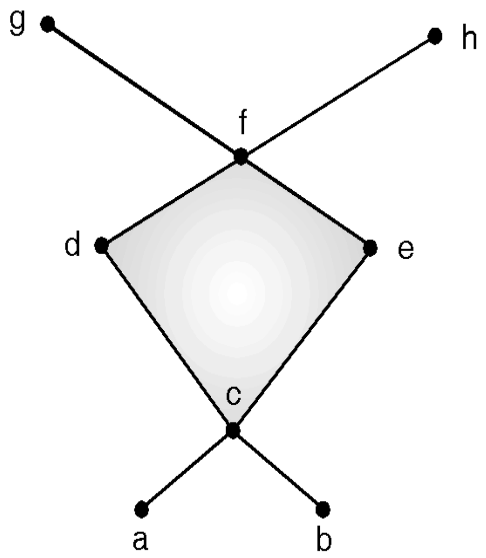
least upper bound is 'c'.

Lower bounds of set B1 is none.

(ii) Upper bounds of set B2 are f, g, and h. There is no least upper bound. Lower bounds of set B2 are c, a, b
Greatest lower bound is ‘c’.

Example

Let A = {a, b, c, d, e, f, g, h} be the poset whose Hasse diagram is shown in Fig.



Find GLB and LUB of B = {c, d, e}.

Solution:

(1)Upper bounds of B are f, g, h.

Least upper bound is f.

(2)Lower bounds of B are c, a, b.

Greatest lower bound is ‘c’

Example

Find the greatest lower bound anc least upper bound of the set (3, 9, 12) and {1, 2, 4, 5, 10} if they exists in the poset (Z+, /) . Where / is relation of diversibility.

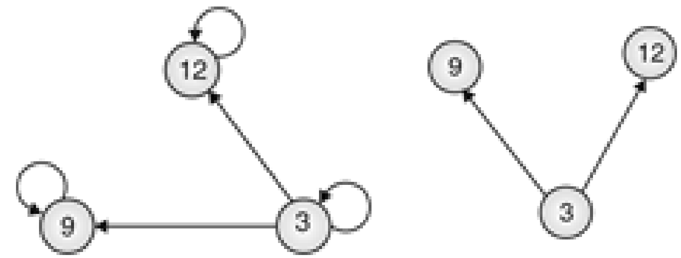
Solution:

(a) A={3, 9, 12}

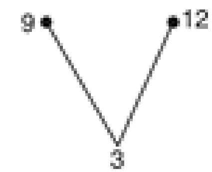
R={(3, 3), (3, 9), (3, 12), (9, 9), (12, 12)}

$$M_R = \begin{array}{c} \\ 3 \\ 9 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 9 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph :



Hasse diagram :



LUB :

	3	9	12
3	3	9	12
9	9	9	-
12	12	-	12

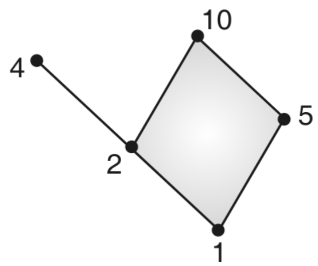
GLB :

	3	9	12
3	3	3	3
9	3	9	3
12	3	3	12

GLB of { 3, 9, 12} =3
LUB of { 3, 9, 12} =36

Example

Hasse Diagram:



(b)A={1, 2, 4, 5, 10 }
R={(1, 1), (1, 2), (1, 4), (1, 5), (1,10), (2, 2), (2, 4), (2, 10), (4, 4), (5, 5), (5, 10), (10, 10)}

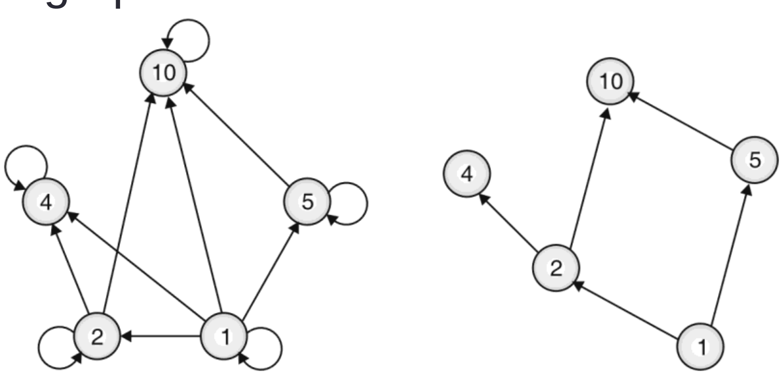
GLB of { 1, 2, 4, 5, 10}=1
LUB of { 1, 2, 4, 5, 10}=20

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 5 & 10 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

LUB :

$$\begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 5 & 10 \end{array} \\ \begin{bmatrix} 1 & 2 & 4 & 5 & 10 \\ 2 & 2 & 4 & 10 & 10 \\ 4 & 4 & 4 & - & - \\ 5 & 10 & - & 5 & 10 \\ 10 & - & 10 & 10 & - \end{bmatrix} \end{array}$$

Digraph:



GLB :

	1	2	4	5	10
1	1	1	1	1	1
2	1	2	2	1	2
4	1	2	4	1	2
5	1	1	1	5	5
10	1	2	2	5	10

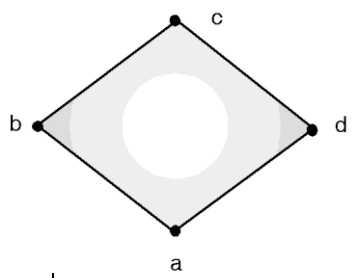
Lattice

A lattice is a poset (L, $\leq$) in which every subset {a, b} consisting of two elements has a least upper bound and a greatest lower bound.

We denote LUB ({a, b}) by a $\vee$ b, and call it the join of a and b. Similarly, we denote GLB ({a, b}) by a $\wedge$ b and call it the meet of a and b.

Example

Ex. 1



LUB :

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

GLB:

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Ex. 2

LUB :

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

GLB:

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Ex. : If L_1 and L_2 are the lattices shown in Fig. 3.68 Draw the Hasse diagram of $L_1 \times L_2$ with the product partial order.

L1  L2

Soln.:

For lattice L_1, Let A={a1,b1}

R_1={$(a_1, a_1), (a_1, b_1), (b_1, b_1)$}

For lattice L_2, Let B={a2,b2}

R_2 ={$(a_2, a_2), (a_2, b_2), (b_2, b_2)$}

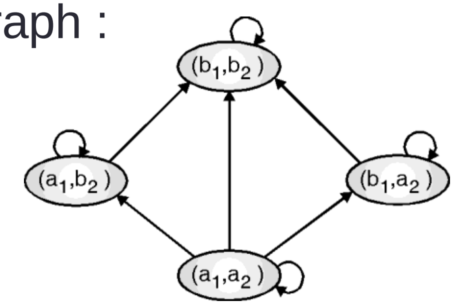
If (A, ≤) and (B, ≤) are posets, then (A × B, ≤) is a poset with partial order ≤ defined by (a, b) ≤ (a′, b′) if a ≤ a′ in A and b ≤ b′ in B.

A × B ={(a1,a2),(a1,b2),(b1,a2),(b1.b2)}

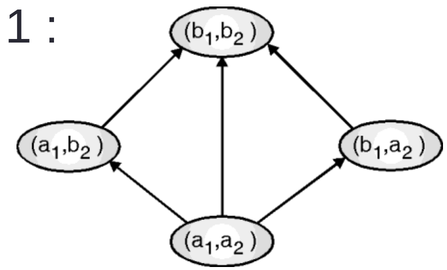
Product partial order relation defined on A × B is as follows :

R_3={$((a_1, a_2), (a_1, a_2)), ((a_1, a_2), (a_1, b_2)), ((a_1, a_2), (b_1, a_2)), ((a_1, a_2), (b_1, b_2)), ((a_1, b_2), (a_1, b_2)), ((a_1, b_2), (b_1, b_2)), ((b_1, a_2), (b_1, a_2)), ((b_1, a_2), (b_1, b_2)), ((b_1, b_2), (b_1, b_2))$}

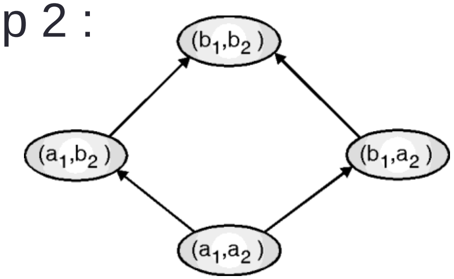
Digraph :



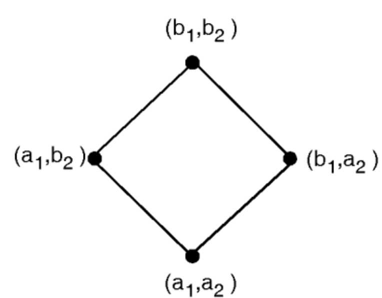
Step 1 :



Step 2 :



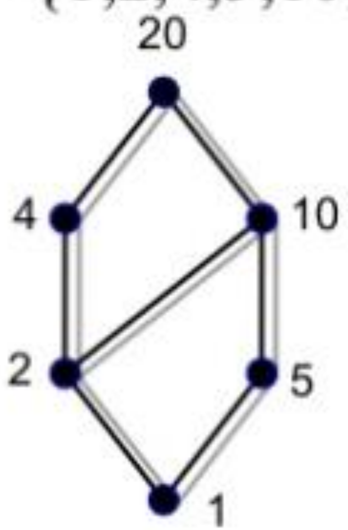
Step 3 :Hasse Diagram of $L_1 \times L_2$

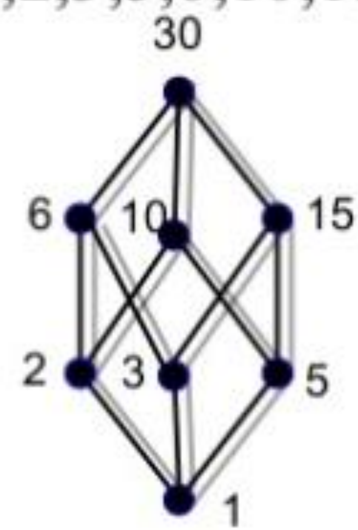


Lattice

Let n be a positive integer and D_n be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility. For instance,

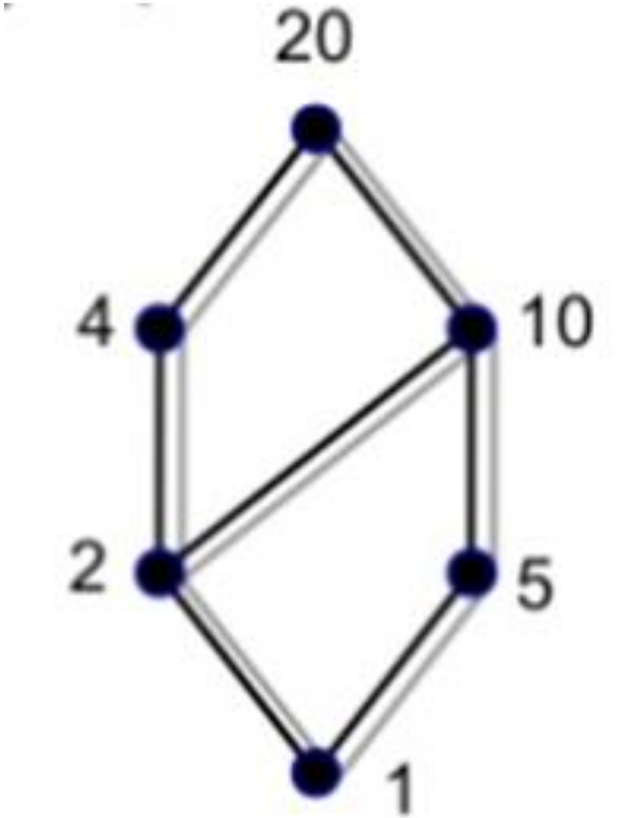
$D_{20} = \{1,2,4,5,10,20\}$ $D_{30} = \{1,2,3,5,6,10,15,20\}$





LUB	1	2	4	5	10	20
1	1	2	4	5	10	20
2	2	2	4	10	10	20
4	4	4	4	20	20	20
5	5	10	20	5	10	20
10	10	10	20	10	10	20
20	20	20	20	20	20	20

GLB	1	2	4	5	10	20
1	1	1	1	1	1	1
2	1	2	2	1	2	2
4	1	2	4	1	2	4
5	1	1	1	5	5	5
10	1	2	2	5	10	10
20	1	2	4	5	10	20



D20 is a Lattice

Dual of a lattice

Let R be a partial order on a set A, and let R^{-1} be the inverse relation of R. Then R^{-1} is also a partial order.

The poset (A, R^{-1}) is galled the **dual** of the poset (A, R).

whenever $(A, \leq)$ is a poset, we use "$\geq$" for the partial order $\leq^{-1}$

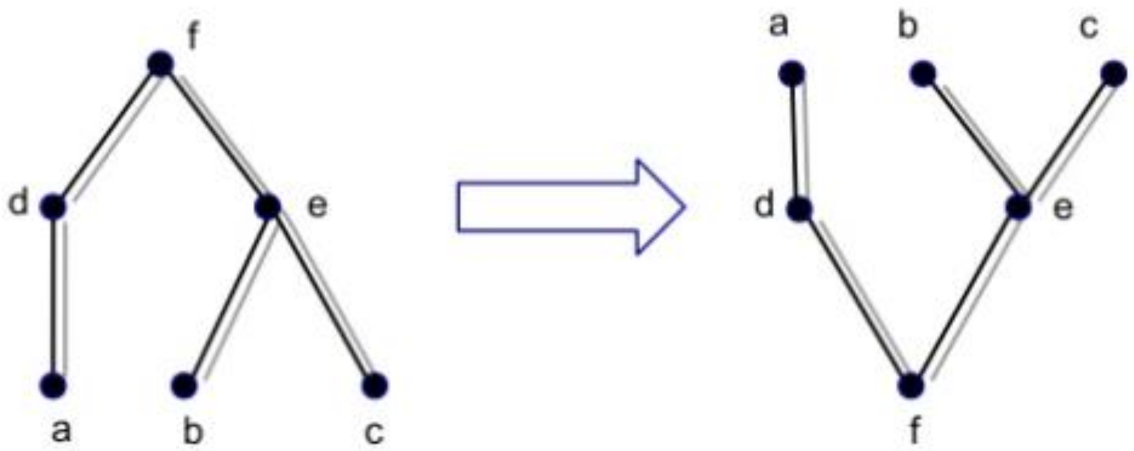
- **Dual of a lattice**: Let $(L, \leq)$ be a lattice, then the $(L, \geq)$ is called dual lattice of $(L, \leq)$.
- **Note**: Dual of dual lattice is original lattice.
- **Note**: In $(L, \leq)$, if $a \vee b = c$; $a \wedge b = d$, then in dual lattice $(L, \geq)$, $a \vee b = d$; $a \wedge b = c$
- **Principle of duality:** If P is a valid statement in a lattice, then the statement obtained by interchanging meet and join everywhere and replacing $\leq$ by $\geq$ is also a valid statement.

Example

Fig. a shows the Hasse diagram of a poset (A, ≤), where

A={a, b, c, d, e, f}

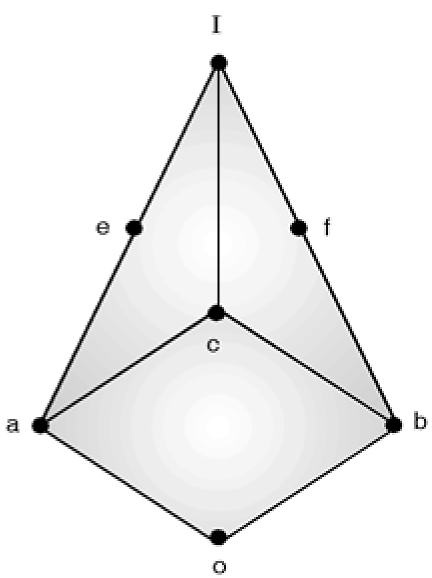
Fig. b shows the Hasse diagram of the dual poset (A, ≥)



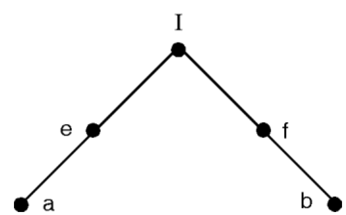
Sub-lattice

Let $(L, \leq)$ be a lattice. A nonempty subset S of L is called a sublattice of L, If $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$.

Lattice:

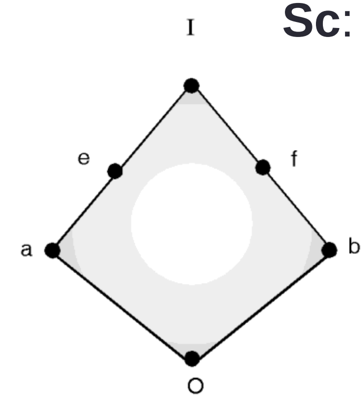


Sb:



The partially ordered subset 'Sb' shown in Fig. is not sub-lattice of L because $a \vee b \notin Sb$ and $a \wedge b \notin S\,b$.

Sc:



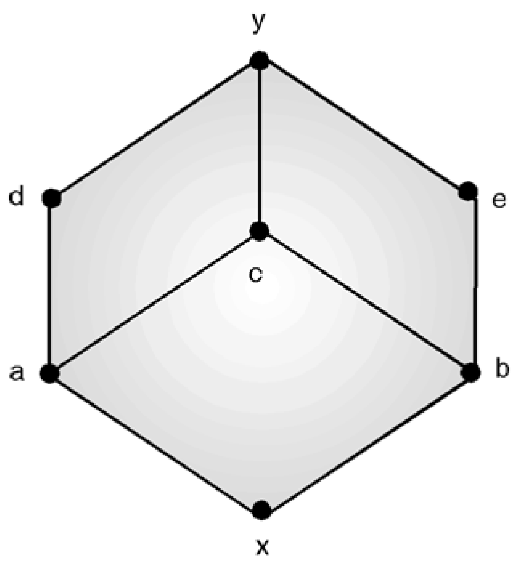
The partially ordered subset 'Sc' shown in Fig. is not sub-lattice of L because $a \vee b \notin Sc$.

Sub-lattice

Consider the lattice L shown in fig. Determine whether or not each of the following is a sublattice of L.

$L_1 = \{x, a, b, y\}$, $L_2 = \{x, a, e, y\}$

$L_3 = \{a, c, d, y\}$, $L_4 = \{x, c, d, y\}$



Now L_1, is not a sublattice since $a \vee b = c$ which does not belong to L_1.

The sets L_2 and L_3 are sublattices.

The subset L_4 is not a sublattice since $c \wedge d = a$ does not belong to L_4.

Properties of Lattices

1. **Idempotent Properties**
   (a) $a \vee a = a$
   (b) $a \wedge a = a$
2. **Commutative Properties**
   (a) $a \vee b = b \vee a$
   (b) $a \wedge b = b \wedge a$
3. **Associative properties**
   (a) $a \vee (b \vee c) = (a \vee b) \vee c$
   (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
4. **Absorption Properties**
   (a) $a \vee (a \vee b) = a$
   (b) $a \wedge (a \vee b) = a$

ISOMORPHIC LATTICES

Definition : Two lattices L and L' are said to be isomorphic if there is a function $f : L \to L'$ such that

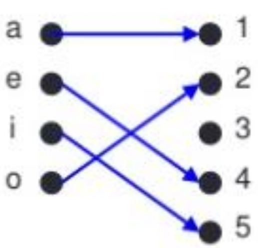
(i) f is one to one

(ii) f is onto (i.e. f is bijection)

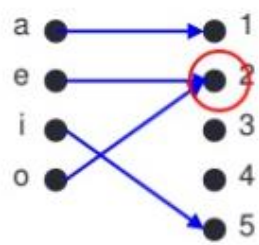
(iii) $f(a \wedge b) = f(a) \wedge f(b)$

(iv) $f(a \vee b) = f(a) \vee f(b)$

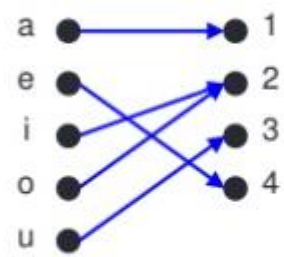
For any elements a, b in L.



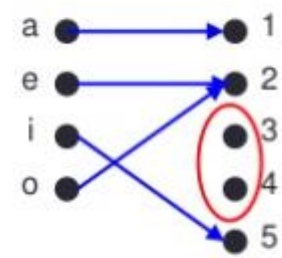
A one-to-one function



A function that is not one-to-one



An onto function



A function that is not onto

Example

Are the two lattices shown in the Fig. isomorphic ?

Soln.: We denote the lattices as L_1 and L_2

$f : L_1 \rightarrow L_2$ as

$f(a)=1$, $f(b)=2$, $f(c)=4$,

$f(d)=3$, $f(e)=5$

Note that the mapping is one-to-one and onto

Also $f(c \wedge d) = f(b) = 2$

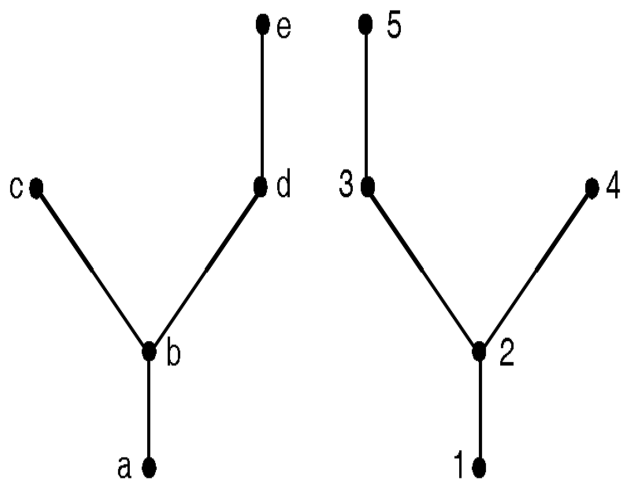
and $f(c)=4$ and $f(d) = 3$

and $f(c) \wedge f(d)=4 \wedge 3 = 2$

$\therefore$ $f(c \wedge d) = f(c) \wedge f(d)$

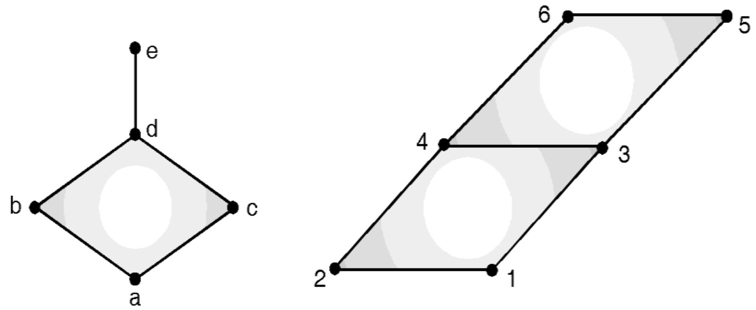
Similarly $f(c \vee d)=f(c) \vee f(d)$

Hence the two lattices are isomorphic



Example

Are the two lattices shown in Fig. isomorphic ?



Solution:

The two lattices are not isomorphic, since the two lattices do not have the same number of elements. Hence, the mapping between two lattices cannot be one to one and onto.

Types of Lattices-Bounded Lattice

Definition :

A lattice L is said to be **bounded** if it has a greatest element **I** and a least element **0**. If L is a bounded lattice, then for all a ϵ A

$$a \vee \mathbf{0} = a, \qquad a \wedge \mathbf{0} = \mathbf{0}$$

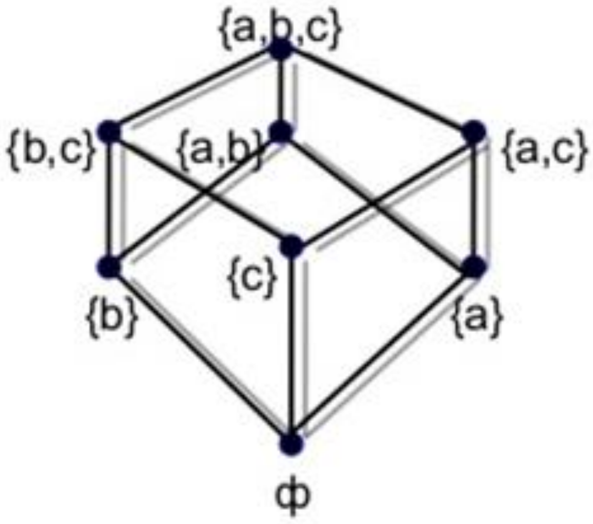
$$a \vee \mathbf{I} = \mathbf{I}, \qquad a \wedge \mathbf{I} = a$$

Types of Lattices-Distributive Lattice

A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If L is not distributive, we say that L is non-distributive.



Example

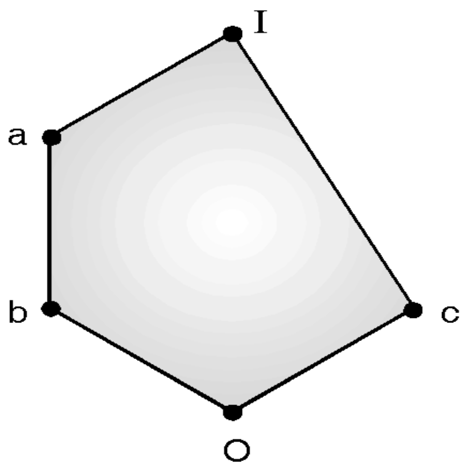


Fig.(a)

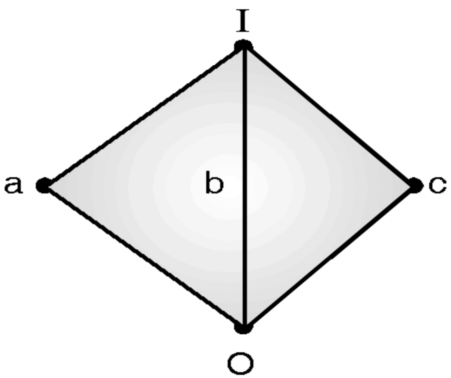


Fig.(b)

(a) We have,

$a \wedge (b \vee c) = a \wedge I = a$

while $(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

So Fig. (a) is non-distributive

(b) Observe that

$a \wedge (b \vee c) = a \wedge I = a$

while $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

So Fig. (b) is non-distributive.

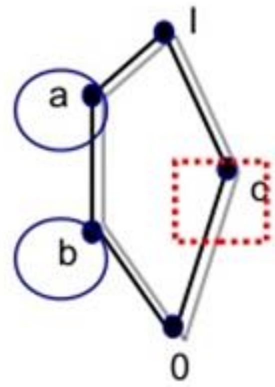
Complemented Lattice

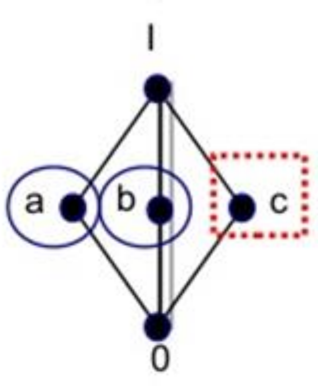
A lattice L is said to be complemented if it is bounded and if every element in L has a complement

Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$.

An element $a' \in L$ is called a complement of a if.

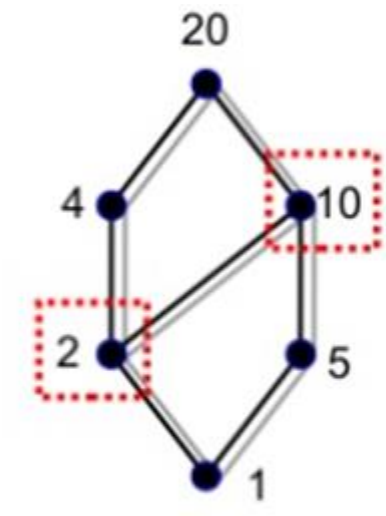
$$a \vee a' = I \qquad \text{and} \qquad a \wedge a' = 0$$





Example

D_{20} is not a complemented lattice



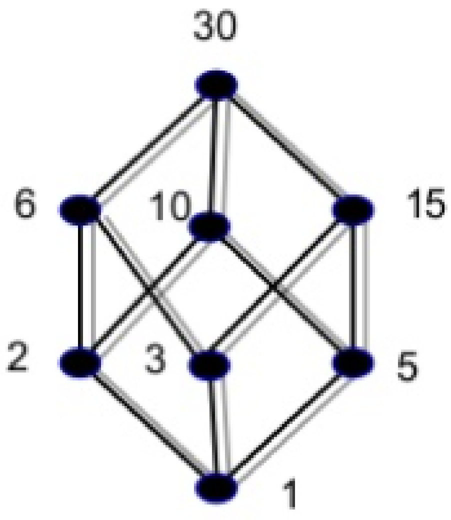
D_{20}

Element	Complement
1	20
2	-
4	5
5	4
10	-
20	1

Example

D_{30} is complemented lattice

Element	Its Complement
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1



D_{30}

Example

Show that in a bounded distributive lattice, if a complement exists, it is unique.

Let a' and a'' be complements of the element.

$a \in L$. Then

$a \vee a' = I$ $\qquad$ $a \vee a'' = I$

$a \wedge a' = 0$ $\qquad$ $a \wedge a'' = 0$

Using the distributive laws, we obtain

$$
\begin{aligned}
a' &= a' \vee 0 \\
&= a' \vee (a \wedge a'') \\
&= (a' \vee a) \wedge (a' \vee a'') \\
&= I \wedge (a' \vee a'') \\
&= a' \vee a''
\end{aligned}
$$

Also,
$$
\begin{aligned}
a'' &= a'' \vee 0 \\
&= a'' \vee (a \wedge a') \\
&= (a'' \vee a) \wedge (a'' \vee a') \\
&= I \wedge (a' \vee a'') \\
&= a' \vee a''
\end{aligned}
$$

Hence, $a' = a''$

Boolean Algebra

A *boolean algebra* is a lattice which contains

1. 2^n elements for any integer $n \geq 0$.
2. A greatest element and a least element.
3. and which is both complemented and distributive.

Ex. Determine whether the following posets are Boolean algebras. Justify your answer.

A = {1, 2, 3, 6} with divisibility

Solution:

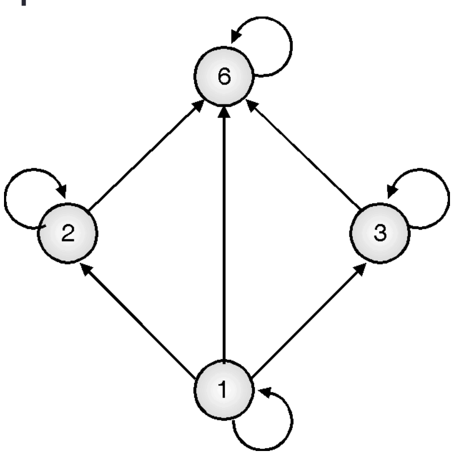
Given set A is {1, 2, 3, 6} and the Partial order relation of divisibility on set A is

R={(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)}

Matrix of the above relation is,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 6 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

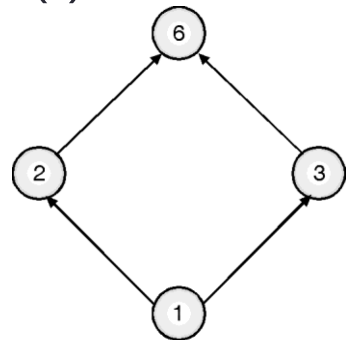
Diagraph is as follows :

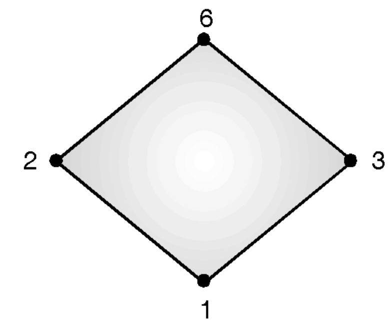


To convert above digraph into Hasse diagram.

(i)Remove cycles

(ii)Remove transitive edge (1, 6)





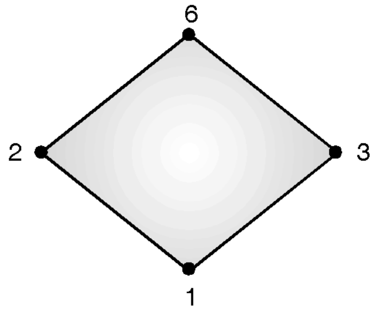
LUB :

∨	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB :

∧	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Every pair of elements in A has a GLB and a LUB. Therefore A is a lattice.



A has a least element (0) = 1

and a greatest element (I) = 6

Number of elements in A is $4=2^2$

$2 \vee 3 = 6$;

$2 \wedge 3 = 1$

$\therefore$ Complement of 2 is 3.

or complement of 3 is 2

A is a complemented lattice,

Also we can show that the operations $\vee$, $\wedge$ are distributive

$2 \vee (2 \wedge 3) = (2 \vee 2) \wedge (2 \vee 3)$

$2 \vee 1 = 2 \wedge 6$

$2 = 2$

$\therefore$ A is a distributive lattice.

A satisfies all requirements of Boolean Algebra.

$\therefore$ A under divisibility is a Boolean Algebra.

Determine if poset represented by each of the Hasse diagrams are lattices. Justify your answer.

Fig. a

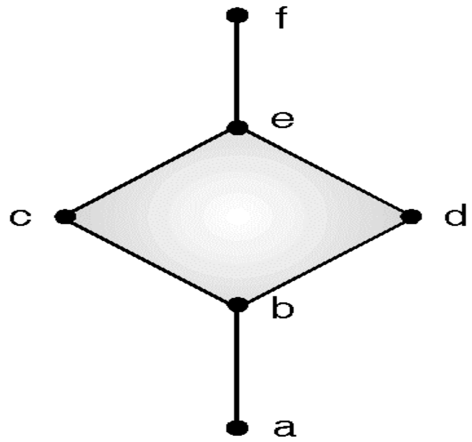


Fig. b

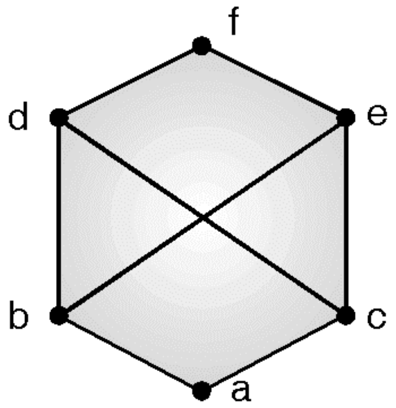


Fig. c