LOGIC-04

2.1 Propositions and logical operations, Truth tables
2.2 Equivalence, Implications
2.3 Laws of logic, Normal Forms
2.4 Predicates and Quantifiers
2.5 Mathematical Induction

LOGIC

- Study of the logic <u>relationships</u> between <u>objects</u>

and

- Basis of all mathematical reasoning and all automated reasoning

Introduction: PL?

- In Propositional Logic, the objects are called propositions
- **Definition**: **A proposition is a statement that is either true or false, but not both**
- We usually denote a proposition by a letter: p, q, r, s, …

Introduction: Proposition

- **Definition**: The value of a proposition is called its truth value; denoted by
  - ***T* or 1 if it is true** or
  - ***F* or 0 if it is false**
- Opinions, interrogative, and imperative are **not propositions**
- **Truth table**

p
0
1

Propositions: Examples

- The following are propositions
  - – Today is Monday *M*
  - – The grass is wet *W*
  - – It is raining *R*
- **The following are not propositions**
  - – C++ is the best language *Opinion*
  - – When is the pretest? *Interrogative*
  - – Do your homework *Imperative*

Logical operations

- Connectives are used to create a compound proposition from two or more propositions
- **Negation** (e.g., $\neg$ a or ! a or ā)
- **AND** or logical **Conjunction** (denoted $\wedge$)
- **OR** or logical **Disjunction** (denoted $\vee$)
- **XOR** or exclusive or (denoted $\oplus$)
- **Imply** on (denoted $\Rightarrow$ or $\rightarrow$)
- Biconditional (denoted $\Leftrightarrow$ or $\leftrightarrow$) IFF

We define the meaning (semantics) of the logical connectives using truth tables

Precedence of Logical Operators

- As in arithmetic, an ordering is imposed on the use of logical operators in compound propositions
- However, it is preferable to use parentheses to disambiguate operators and facilitate readability

$$\neg p \vee q \wedge \neg r \equiv (\neg p) \vee (q \wedge (\neg r))$$

- To avoid unnecessary parenthesis, the following precedence hold:
  1. Negation ($\neg$)
  2. Conjunction ($\wedge$)
  3. Disjunction ($\vee$)
  4. Implication ($\rightarrow$)
  5. Biconditional ($\leftrightarrow$)

Logical Connective: **Negation**

- $\neg p$, the negation of a proposition p, is also a proposition
- Examples:
  - Today is not Monday
  - It is not the case that today is Monday, etc.
- **Truth table**

p	$\neg p$
0	1
1	0

Logical Connective: **Logical AND**

- The logical connective And is true only when both of the propositions are true. It is also called a <u>conjunction</u>
- Examples
  - It is raining and it is warm
  - (2+3=5) and (1<2)
- Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Connective: **Logical OR**

- The logical <u>disjunction</u>, or logical OR, is true if one or both of the propositions are true.
- Examples
  - It is raining or it is the second lecture
  - $(2+2=5) \vee (1<2)$
  - You may have cake or ice cream
- **Truth table**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Connective: **Exclusive Or**

- The exclusive OR, or XOR, of two propositions is true when exactly one of the propositions is true and the other one is false
- Example
  - The circuit is either ON or OFF but not both
- Truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Connective: **Biconditional** (1)

- **Definition:** The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values. It is false otherwise.
- **Truth table**

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical Connective: Biconditional (2)

- The biconditional $p \leftrightarrow q$ can be equivalently read as
  - p if and only if q
  - p is a necessary and sufficient condition for q
  - if p then q, and conversely
  - p iff q (Note typo in textbook, page 9, line 3)
- Examples
  - $x>0$ **if and only** if x^2 is positive
  - The alarm goes off **iff** a burglar breaks in
  - You may have pudding **iff** you eat your meat

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Examples :

- (i) An integer is even if and only if it is divisible by 2.
- (ii) A right angled triangle is isosceles if and only if the other two angles are each equal to forty-five degrees.
- (iii) Two lines are parallel if and only if they have the same slope

Logical Connective: **Implication** (1)

- **Definition:** Let *p* and *q* be two propositions.

The implication $\boldsymbol{p \rightarrow q}$ is **a relationship between two propositions in which the second is a logical consequence of the first**

- *p* is called the hypothesis(If I give you 1 million $)
- *q* is called the conclusion, consequence(then you will become a millionaire)

Note that this is logically equivalent to ¬p V q

Logical Connective: Implication

- The implication of $p \rightarrow q$ can be also read as
  - If p then q
  - p implies q
  - If p, q
  - p only if q
  - q if p
  - q when p
  - q whenever p
  - q follows from p
  - p is a sufficient condition for q (p is sufficient for q)
  - q is a necessary condition for p (q is necessary for p)

Logical Connective: Implication

- Examples
  - If you buy you air ticket in advance, it is cheaper.
  - If x is an integer, then $x^2 \geq 0$.
  - If it rains, then grass gets wet.
  - If 2+2=5, then all unicorns are pink.

Truth Table

If you score 85% or above in this class(P), then you will get an A (Q).

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Let **p** denote Peter is rich, **q** denote Peter is happy.
Write symbolic form for the following

1. Peter is poor but happy
2. **$\sim p \wedge q$**
3. Peter is neither rich nor happy
4. **$\sim p \wedge \sim q$**
5. Peter is either rich or unhappy
6. **$p \vee \sim q$**

Write the following statements in symbolic form :

p :I will study discrete structures.
q :I will go to a movie.
r : I am in a good mood.

1. If I am not in a good mood, then I will go to a movie.

2. I will not go to a movie and I will study discrete structures.

3. I will go to a movie only if I will not study discrete structures.

4. I will not study discrete structures, then I am not in a good mood.

p :I will study discrete structures.

q :I will go to a movie.

r : I am in a good mood.

Write the following statements in symbolic form :

- If I am not in a good mood, then I will go to a movie.
- **$\sim r \rightarrow q$**
- I will not go to a movie and I will study discrete structures.
- **$\sim q \wedge p$**
- I will go to a movie only if I will not study discrete structures.
- **$\sim p \rightarrow q$**
- I will not study discrete structures, then I am not in a good mood.
- **$\sim p \rightarrow \sim r$**

Write logical/conditional propositions

1. There is an error in the program or the data is wrong.
2. If Peter works hard then he will pass the exam.
3. Farmers will face hardship if the dry spell continues.
4. Unless I reach the station on time , I will miss the train.

Converse, Inverse, Contrapositive

- Consider the proposition $p \rightarrow q$ (Conditional If….. then)
  - Its **<u>converse</u>** is the proposition $\mathbf{q \rightarrow p}$
  - Its **<u>contrapositive</u>** is the proposition $\mathbf{\neg q \rightarrow \neg p}$
  - Its **<u>inverse</u>** is the proposition $\mathbf{\neg p \rightarrow \neg q}$

- State the converse, inverse and contrapositive of the following.
- (i) If it is cold then he wears hat.
- Let p: It is cold , q: He wears hat.
- **Converse** $(q \rightarrow p)$ **:** If he wears hat then it is cold.
- **Contrapositive** $(\sim q \rightarrow \sim p)$ **:** If he does not wear hat, then it is not cold.
- **Inverse** $(\sim p \rightarrow \sim q)$ **:** If it is not cold then he does not wear hat.
- (ii) If integer is multiple of 2, then it is even.
- **Converse** $(q \rightarrow p)$ **:** If integer is even, then it is multiple of 2.
- **Inverse** $(\sim p \rightarrow \sim q)$: If integer is not multiple of 2 then it is not even.
- **Contrapositive** $(\sim q \rightarrow \sim p)$ **:** If integer is not even, then it is not multiple of 2.

1. Consider

P:You stay in Mumbai;

Q: You stay in Taj

Determine Converse, Contrapositive and Inverse for

“If you stay in Mumbai , you stay in Taj”

2. Write down the English sentences for converse and contrapositive of : **“If 250 is divisible by 4 then 250 is an even number “**

Let '**a**' be the proposition **'high speed driving is dangerous**' and '**b**' be the proposition **'Rajesh was a wise man.'**

Write down the meaning of the following proposition.

- 1. $a \wedge b$
- 2. $\sim a \wedge b$
- 3. $\sim (a \wedge b)$
- 4. $(a \wedge b) \vee (\sim a \wedge \sim b)$
- **Soln. :**
- **1.** $\mathbf{a \wedge b}$ **:** High speed driving is dangerous and Rajesh was a wise man.
- **2.** $\mathbf{\sim a \wedge b}$ **:** High speed driving is not dangerous and Rajesh is a wise man.
- **3.** $\mathbf{\sim (a \wedge b)}$ **:** Either high speed driving is not dangerous or Rajesh is not a wise man.
- **4.** $\mathbf{(a \wedge b) \vee (\sim a \wedge \sim b)}$ **:** High speed driving is dangerous and Rajesh was a wise man or neither high speed driving is dangerous nor Rajesh is a wise man.

- Express the proposition 'Either my program runs and it contains no bugs, or my program contains bugs' in symbolic form.
- **Soln. :**
- Let p : My program runs.
- q : My program contains bugs.
- The proposition can be written in symbolic form as,

$$\mathbf{(\, p \wedge \sim q \,) \vee q}$$

How can this English sentence be translated into a logical expression?

"You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

- **Soln: Let q : You can ride the roller coaster.**
- **r : You are under 4 feet tall.**
- **s : You are older than 16 years old.**
- **The sentence can be translated into:**
- $(r \land \neg s) \rightarrow \neg q$

Usefulness of Logic

- Logic is more precise than natural language
  - You may have cake or ice cream.
    - Can I have both?
  - If you buy your air ticket in advance, it is cheaper.
    - Are there or not cheap last-minute tickets?
- For this reason, logic is used for hardware and software <u>specification</u>
  - Given a set of logic statements,
  - One can decide whether or not they are <u>satisfiable</u> (i.e., consistent), although this is a costly process…

Terminology:
Tautology, Contradictions, Contingencies

- Definitions
  - A compound proposition that is always **true**, no matter what the truth values of the propositions that occur in it is called a **TAUTOLOGY**
  - A compound proposition that is always **false** is called a **CONTRADICTION**
  - A proposition that is neither a tautology nor a contradiction is a **CONTINGENCY**
- Examples
  - A simple tautology is $p \vee \neg p$
  - A simple contradiction is $p \wedge \neg p$

Truth Tables

- Truth tables are used to show/define the relationships between the truth values of
  - the individual propositions and
  - the compound propositions based on them

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Logical Equivalences: Example 1

- Are the propositions $(p \rightarrow q)$ and $(\neg p \vee q)$ logically equivalent?
- To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0			
0	1			
1	0			
1	1			

- The two columns in the truth table are identical, thus we conclude that

$$(p \rightarrow q) \equiv (\neg p \vee q)$$

Constructing Truth Tables

- Construct the truth table for the following compound proposition ~ P ∧ (P →Q)

$\sim P \wedge (P \rightarrow Q)$.

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

. Show that $(P \rightarrow Q) \vee (Q \rightarrow P)$ is a tautology.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Prove $(A \vee B) \wedge [(\neg A) \wedge (\neg B)]$ is a contradiction

Prove $(A \vee B) \wedge (\neg A)$ a contingency

- Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Logical Equivalences: Exercise 25 from Rosen

- Show that $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \vee (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$

LAWS OF LOGIC

Implication law:

$P \rightarrow Q = \neg P \vee Q$

Commutative laws

$p \wedge q \Leftrightarrow q \wedge p$

$p \vee q \Leftrightarrow q \vee p$

Associative laws

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

Inverse laws

$p \wedge \neg p \Leftrightarrow F$

$p \vee \neg p \Leftrightarrow T$

Laws of Logic

Distributive laws

$p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$

$p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$

Idempotent laws

$p \land p \Leftrightarrow p$

$p \lor p \Leftrightarrow p$

Identity laws

$p \land T \Leftrightarrow p$

$p \lor F \Leftrightarrow p$

Laws of Logic

Domination laws

$p \land F \Leftrightarrow F$

$p \lor T \Leftrightarrow T$

Absorption law

$p \land (p \lor q) \Leftrightarrow p$

$p \lor (p \land q) \Leftrightarrow p$

De Morgan Law:

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

Table of Logical Equivalences

Commutative	$p \wedge q \iff q \wedge p$	$p \vee q \iff q \vee p$
Associative	$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	$(p \vee q) \vee r \iff p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
Negation	$p \vee \sim p \iff T$	$p \wedge \sim p \iff F$
Double Negative	$\sim (\sim p) \iff p$	
Idempotent	$p \wedge p \iff p$	$p \vee p \iff p$
Universal Bound	$p \vee T \iff T$	$p \wedge F \iff F$
De Morgan's	$\sim (p \wedge q) \iff (\sim p) \vee (\sim q)$	$\sim (p \vee q) \iff (\sim p) \wedge (\sim q)$
Absorption	$p \vee (p \wedge q) \iff p$	$p \wedge (p \vee q) \iff p$
Conditional	$(p \implies q) \iff (\sim p \vee q)$	$\sim (p \implies q) \iff (p \wedge \sim q)$

Quantifiers

- **Predicates**- Quantifier is **used to quantify the variable of predicates**

  Eg : x is a city in India ---------------------------- P(x)

  x is the father of y -----------------------------P(x , y)

  x + y > = z is denoted by ----------------------P(x, y, z)

  – There are two types of quantifier in predicate logic –

  – **Universal Quantifier and Existential Quantifier**

Universal **quantifier** "**for all, for every, for each** "

is **"∀"**,

Existential **quantifier**" **there exists** " is "∃"

$\forall\ a\ \exists\ b\ P(a, b)$

where $P(a, b)$ denotes $a + b = 0$

$\forall\ a\ \forall\ b\ \forall\ c\ P(a, b, c)$

where $P(a, b, c)$ denotes $a + (b + c) = (a + b) + c$

Note – $\forall\ a\ \exists\ b\ P(x, y) \neq \exists\ a\ \forall\ b\ P(x, y)$

Quantifier

If M(x) is "x is man"

C(x) is "x is clever"

Translate the following statements into English.

$$\text{(i) } \exists\, x\, (M(x) \rightarrow C(x))$$

$$\text{(ii) } \forall\, x\, (M(x) \wedge C(x))$$

- **Soln. :**
- (i) There exists a man who is clever.
- (ii) For all men x is man and x is clever.

Write the following two propositions in symbols.

Let p(x,y) denote the predicate 'y = x + 1'.

(i)'For every number x there is a number y such that y = x + 1.'

$$\forall x \exists y P(x, y)$$

(ii)'There is a number y such that, for every number x, y = x + 1.'

$$\exists y \forall x P(x,y)$$

Write English sentences for the following

1. $\forall x \exists y R(x, y)$
2. $\exists x \forall y R(x, y)$
3. $\forall x (\sim Q(x))$
4. $\exists x (\sim P(x))$
5. $\forall x P(x)$

where

P(x) : x is even

Q(x) : x is prime nos

R(x, y): x + y is even

NORMAL FORMS(complex form of variables)

- **Conjunctive Normal Form- CNF**
  - Expression $(x_1 \vee x_2 \vee x_3) \wedge (\sim x_1 \vee \sim x_2 \vee \sim x_3)$

  Eg: $(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)$

- **Disjunctive Normal Form- DNF**
  - Expression $(x_1 \wedge x_2 \wedge x_3) \vee (\sim x_1 \wedge \sim x_2 \wedge \sim x_3)$

  Eg: $(A \wedge B) \vee (A \wedge C) \vee (B \wedge C \wedge D)$

Method to construct DNF

1. Construct a truth table for the proposition.
2. Use the rows of the truth table where the proposition is **True to construct minterms.**
   - If the variable is true, use the propositional variable in the minterm
   - **If a variable is false, use the negation** of the variable in the minterm

1. Connect the minterms with $\vee$'s.

How to find the DNF of $(p \vee q) \to \neg r$

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \to \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

$(p \vee q) \to \neg r \Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee$
$(\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

Conjunctive Normal Form$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$

Truth table:

p	q	r	$p \Leftrightarrow q$	$\neg p \wedge r$	$(p \Leftrightarrow q) \Rightarrow (\neg p \wedge r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	1	0	0
1	1	1	1	0	0

Solution: (full) conjunctive normal form (FCNF) of the formula is $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

Find DNF of $(\sim p \rightarrow r) \wedge (p \Leftrightarrow q)$

Find CNF of $(p \Rightarrow q) \Rightarrow r$

$(p \wedge \neg q) \vee q \Leftrightarrow p \vee q$

$(p \wedge \neg q) \vee q$	Left-Hand Statement
$\Leftrightarrow q \vee (p \wedge \neg q)$	Commutative
$\Leftrightarrow (q \vee p) \wedge (q \vee \neg q)$	Distributive
$\Leftrightarrow (q \vee p) \wedge T$	Negation
$\Leftrightarrow q \vee p$	Identity
$\Leftrightarrow p \vee q$	Commutative

Prove: $p \rightarrow p \vee q$ is a tautology

$p \rightarrow p \vee q$	
$\Leftrightarrow \neg p \vee (p \vee q)$	Implication Equivalence
$\Leftrightarrow (\neg p \vee p) \vee q$	Associative
$\Leftrightarrow (p \vee \neg p) \vee q$	Commutative
$\Leftrightarrow T \vee q$	Negation
$\Leftrightarrow q \vee T$	Commutative
$\Leftrightarrow T$	Domination

Obtain CNF $(p \wedge q) \vee (p \wedge \neg q)$

$(p \wedge q) \vee (p \wedge \neg q)$

$\equiv ((p \wedge q) \vee p) \wedge ((p \wedge q) \vee \neg q)$

$\equiv ((p \vee p) \wedge (q \vee p)) \wedge ((p \vee \neg q) \wedge (q \vee \neg q))$

Obtain DNF $\sim(p \rightarrow (q \wedge r))$

Solution :- $(p \wedge \sim q) \vee (p \wedge \sim r)$

Use **MATHEMATICAL INDUCTION** to prove that

1 + 2 + 3 + ... + n = n (n + 1) / 2 for all positive integers n.

Let the statement P (n) be 1 + 2 + 3 + ... + n = n (n + 1) / 2

STEP 1. Basis: We first show that p (1) is true.

Left Side = 1

Right Side = 1 (1 + 1) / 2 = 1

Both sides of the statement are equal hence p (1) is true.

STEP 2-Inductive step : We now assume that p (k) is true

1 + 2 + 3 + ... + k = k (k + 1) / 2

STEP 3:Inductive hypothesis , Show that p (k + 1) is true by adding k + 1 to both sides of the above statement

1 + 2 + 3 + ... + k + (k + 1) = k (k + 1) / 2 + (k + 1)

= (k + 1)(k / 2 + 1)

= (k + 1)(k + 2) / 2

The last statement may be written as

1 + 2 + 3 + ... + k + (k + 1) = (k + 1)(k + 2) / 2 Which is the statement p(k + 1).

Exercises- Prove each of the following by Mathematical Induction

For n positive integers n, solve the following

→ $1 + 5 + 9 + \ldots\ldots + (4n - 3) = n(2n - 1)$

→ $1^2 + 2^2 + \ldots + n^2 = (n)(n + 1)(2n + 1) / 6$

→ $\mathbf{1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4}$

Prove that for any positive integer number n , $\mathbf{n^3 + 2n}$ is divisible by 3

Statement P (n) is defined by $n^3 + 2n$ is divisible by 3

STEP 1: We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$k^3 + 2k$ is divisible by 3

is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

- Prove that $5^{n} -1$ is divisble by 4 for all $n>=1$

$5^{1} -1= 4$; $P(1)$ is true

$5^{k} -1$ assume to be true

$= 5^{k+1} -1$

$= 5^{k} * 5 - 5 + 4$

$5 (5^{k} -1) + 4$

Mathematical Induction

Ex. 1 : Prove by induction :

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$ for all natural number values of n.

Soln. : Let P(n) be the statement :

$$1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}$$

(i) Basis of induction :

for n = 1,

$$P(1) : 1 = \frac{1\,(2)}{2} = 1$$

Hence P(1) is true.

(ii) Induction step :

Assume P(k) is true,

$$P(k) : 1 + 2 + 3 + \ldots + k = \frac{k\,(k+1)}{2} \quad \ldots(i)$$

(This assumption is called the induction hypothesis)

Prove P(k + 1) is also true.

$$P(k+1) : 1 + 2 + 3 + \ldots + k + (k+1)$$

$$= \frac{(k+1)\,[\,(k+1)+1]}{2} \quad \ldots(ii)$$

$$= \frac{(k+1)\,(k+2)}{2}$$

Using equation (i)

$$\frac{k\,(k+1)}{2} + (k+1) = \frac{(k+1)\,(k+2)}{2}$$

$$\frac{(k+1)\,(k+2)}{2} = \frac{(k+1)\,(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also ture. Therefore P(n) is true for all natural number values of n.