# *Input modeling*

# Contents

# Purpose & Overview

- Input models provide the driving force for a simulation model.
- The quality of the output is no better than the quality of inputs.
- In this chapter, we will discuss the 4 steps of input model development:

(1) Collect data from the real system

(2) Identify a probability distribution to represent the input process

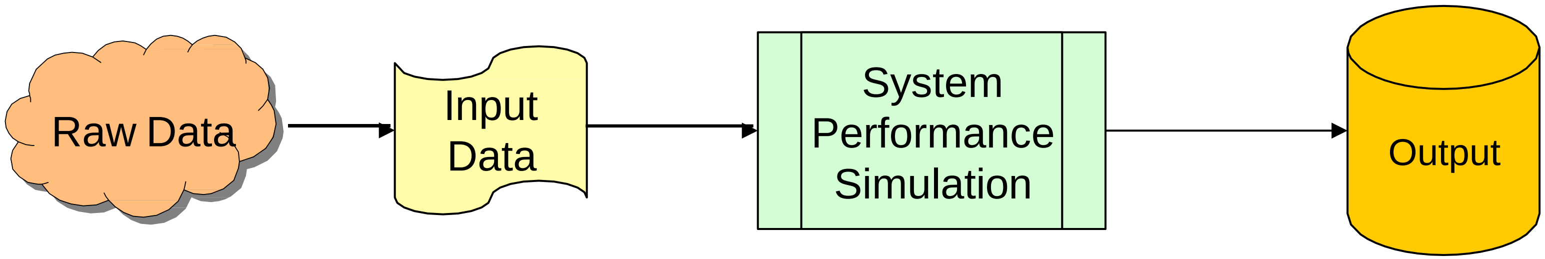
(3) Choose parameters for the distribution

(4) Evaluate the chosen distribution and parameters for goodness of fit

# Data Collection

# Data Collection

- One of the biggest tasks in solving a real problem
  - GIGO: Garbage-In-Garbage-Out




- Even when model structure is valid simulation results can be misleading, if the input data is
  - inaccurately collected
  - inappropriately analyzed
  - not representative of the environment

# Data Collection

- Suggestions that may enhance and facilitate data collection:
  - **Plan ahead:** begin by a practice or pre-observing session, watch for unusual circumstances
  - **Analyze the data as it is being collected:** check adequacy
  - **Combine homogeneous data sets:** successive time periods, during the same time period on successive days
  - **Be aware of data censoring:** the quantity is not observed in its entirety, danger of leaving out long process times
  - **Check for relationship between variables** (scatter diagram)
  - **Check for autocorrelation**
  - **Collect input data, not performance data**

# Identifying the Distribution

Histograms

# Histograms

- A frequency distribution or histogram is useful in determining the shape of a distribution

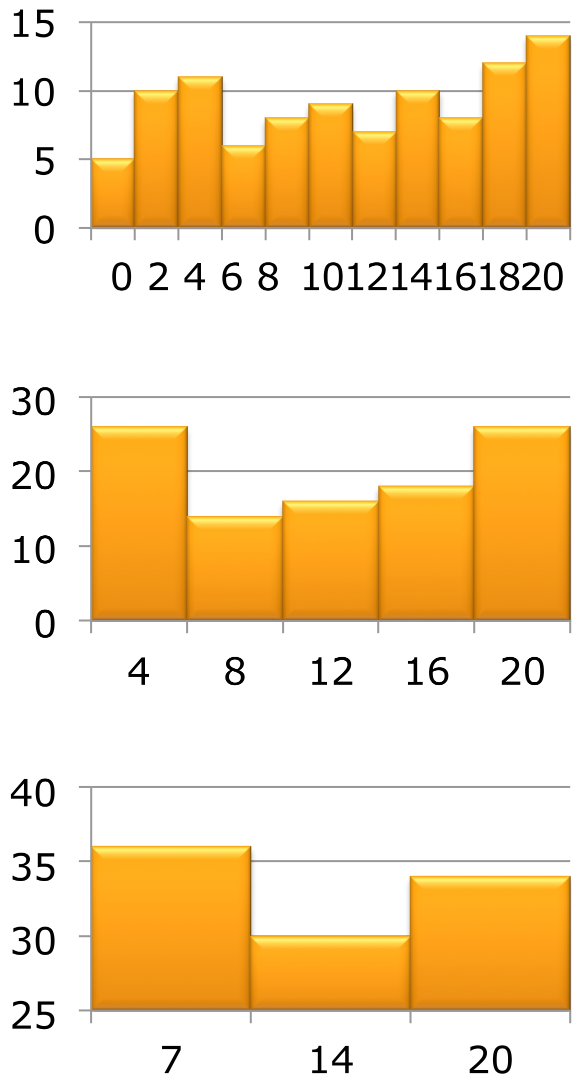
- The number of class intervals depends on:
  - The number of observations
  - The dispersion of the data
  - Suggested number of intervals: the square root of the sample size

- For continuous data:
  - Corresponds to the probability density function (pdf) of a theoretical distribution
- For discrete data:
  - Corresponds to the probability mass function (pmf)

- If few data points are available
  - combine adjacent cells to eliminate the ragged appearance of the histogram

# Histograms

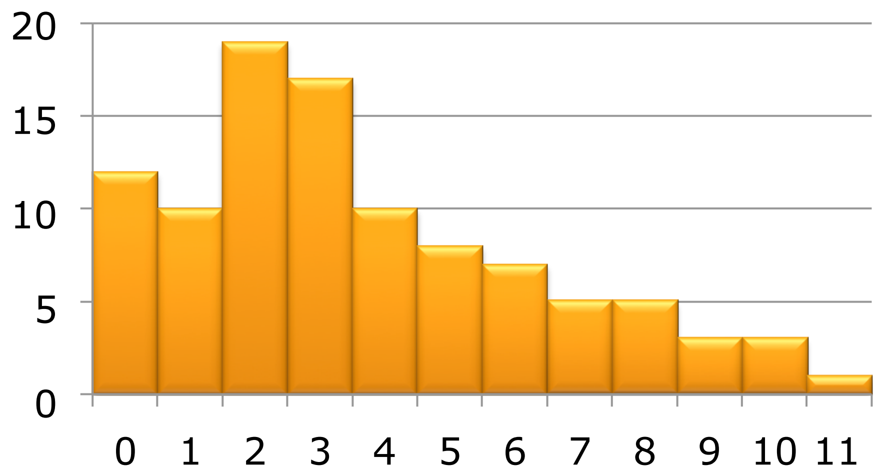
- Same data with different interval sizes



# Histograms: Example

- Vehicle Arrival Example: Number of vehicles arriving at an intersection between 7 am and 7:05 am was monitored for 100 random workdays.
- There are ample data, so the histogram may have a cell for each possible value in the data range

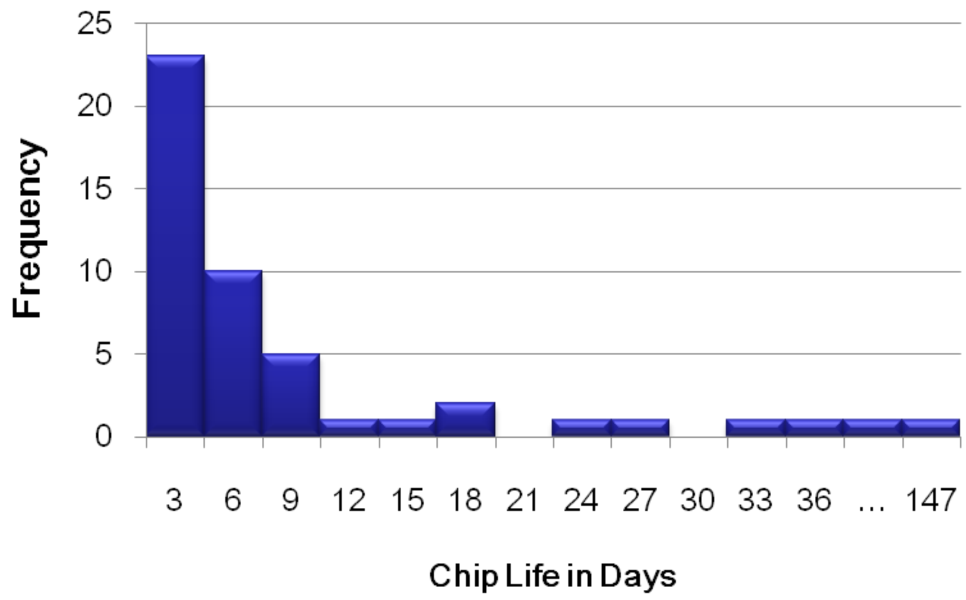
| Arrivals per Period | Frequency |
|---|---|
| 0 | 12 |
| 1 | 10 |
| 2 | 19 |
| 3 | 17 |
| 4 | 10 |
| 5 | 8 |
| 6 | 7 |
| 7 | 5 |
| 8 | 5 |
| 9 | 3 |
| 10 | 3 |
| 11 | 1 |



# Histograms: Example

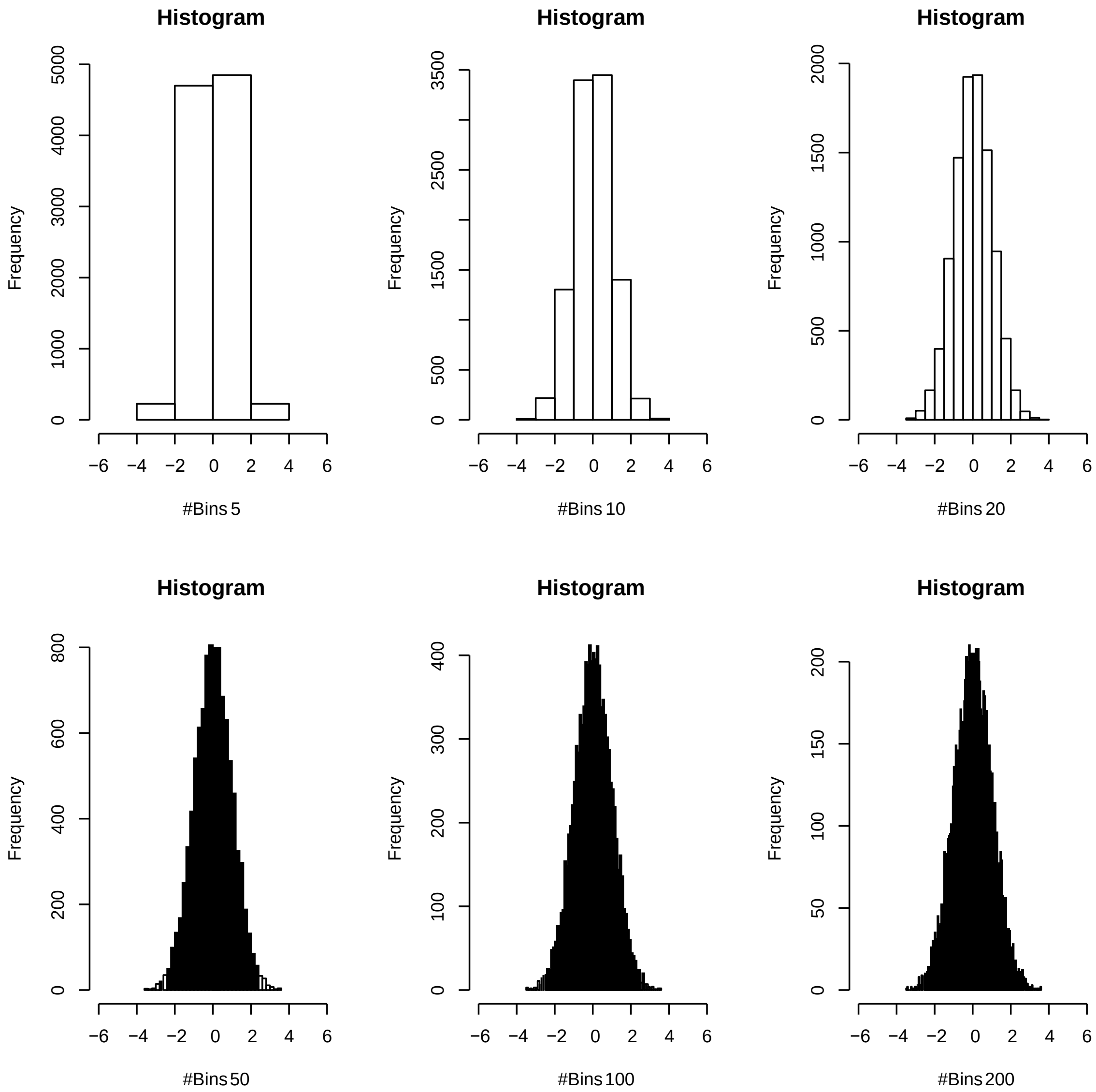
- Life tests were performed on electronic components at 1.5 times the nominal voltage, and their lifetime was recorded

| Component Life | Frequency |
|---|---|
| $0 \le x < 3$ | 23 |
| $3 \le x < 6$ | 10 |
| $6 \le x < 9$ | 5 |
| $9 \le x < 12$ | 1 |
| $12 \le x < 15$ | 1 |
| … | |
| $42 \le x < 45$ | 1 |
| … | |
| $144 \le x < 147$ | 1 |



# Histograms: Example

- Sample size 10000
- Histograms with different numbers of bins



# Histograms: Example

**S̲tanford U̲niversity M̲obile A̲ctivity Tra̲ces (SUMATRA)**

- Target community: cellular network research community
- Traces contain mobility as well as connection information

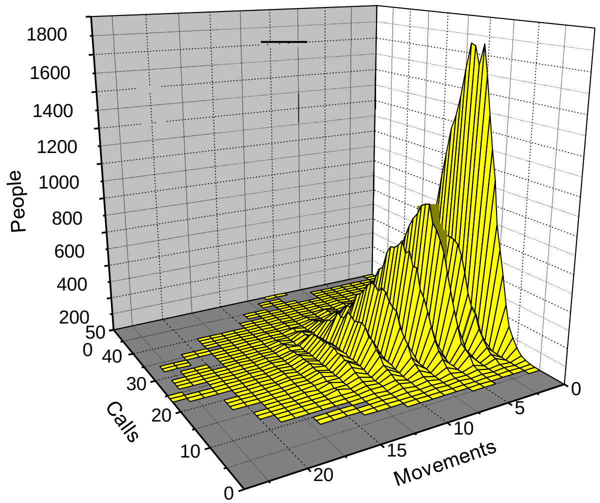
- Available traces
  - SULAWESI (S.U. Local Area Wireless Environment Signaling Information)
  - BALI (B̲ay A̲rea L̲ocation I̲nformation)

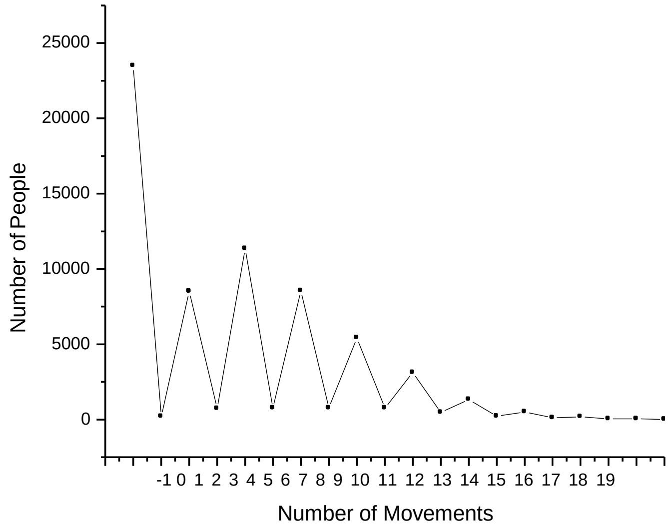
- BALI Characteristics
  - San Francisco Bay Area
  - Trace length: 24 hour
  - Number of cells: 90
  - Persons per cell: 1100
  - Persons at all: 99.000
  - Active persons: 66.550
  - Move events: 243.951
  - Call events: 1.570.807


- Question: How to transform the BALI information so that it is usable with a network simulator, e.g., ns-2?
  - Node number as well as connection number is too high for ns-2

# Histograms: Example

- Analysis of the BALI Trace
  - Goal: Reduce the amount of data by identifying user groups
- User group
  - Between 2 local minima
  - Communication characteristic is kept in the group
  - A user represents a group
- Groups with different mobility characteristics
  - Intra- and inter group communication
- Interesting characteristic
  - Number of people with odd number movements is negligible!



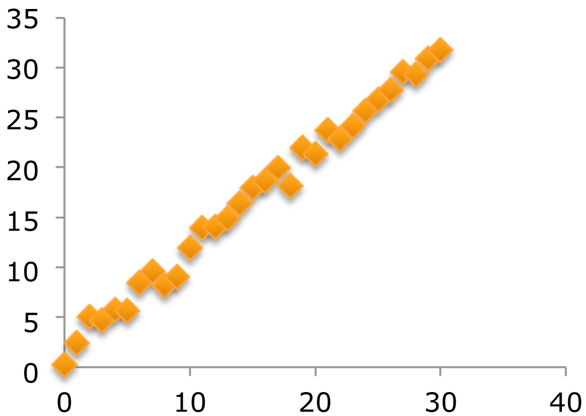



## Identifying the Distribution

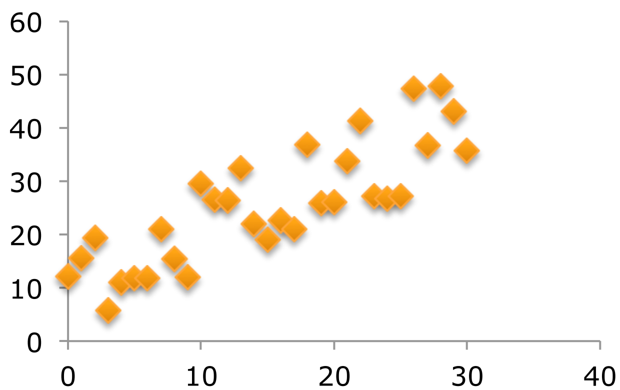
Scatter diagrams

# Scatter Diagrams

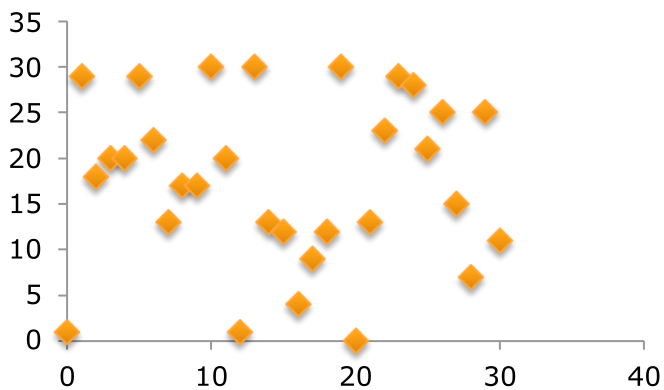
- A scatter diagram is a quality tool that can show the relationship between paired data
  - Random Variable $X$ = Data 1
  - Random Variable $Y$ = Data 2
  - Draw random variable $X$ on the $x$-axis and $Y$ on the $y$-axis




Strong Correlation



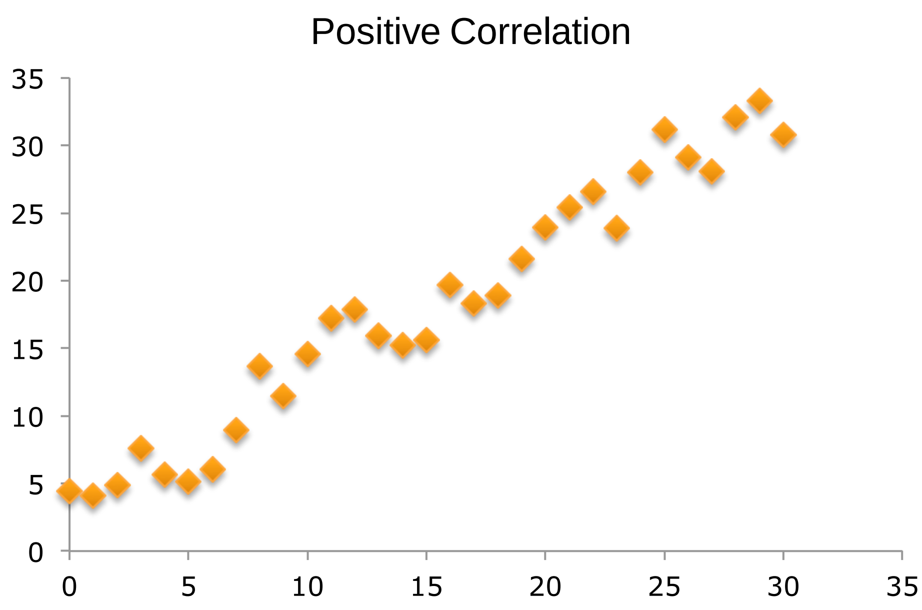

Moderate Correlation

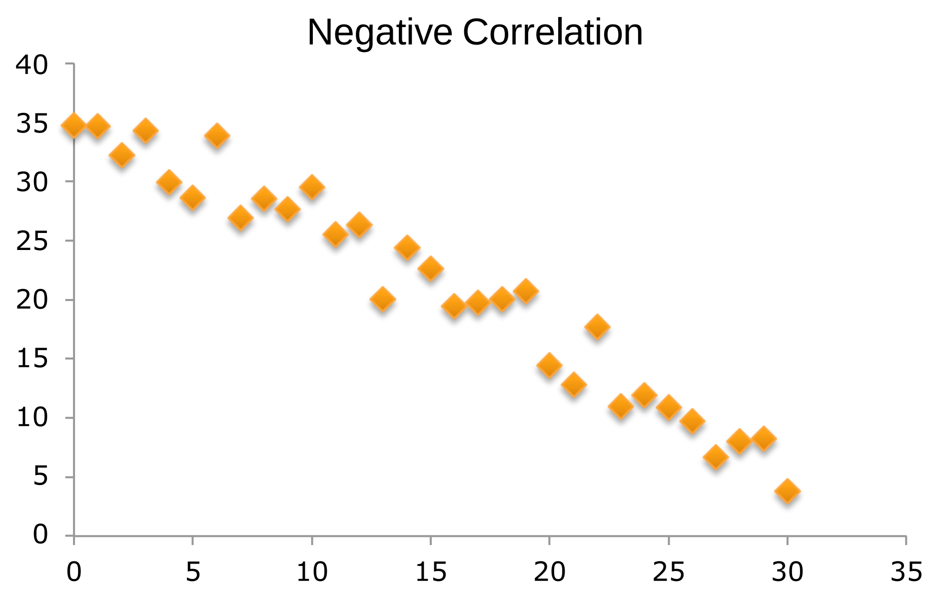



No Correlation

# Scatter Diagrams

- Linear relationship
  - Correlation: Measures how well data line up
  - Slope: Measures the steepness of the data
  - Direction
  - Y intercept



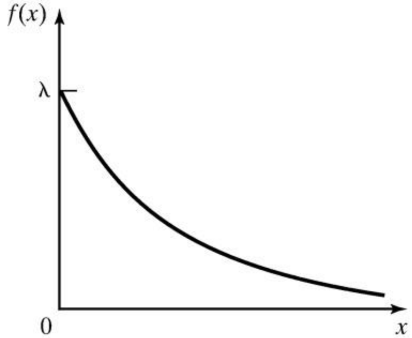



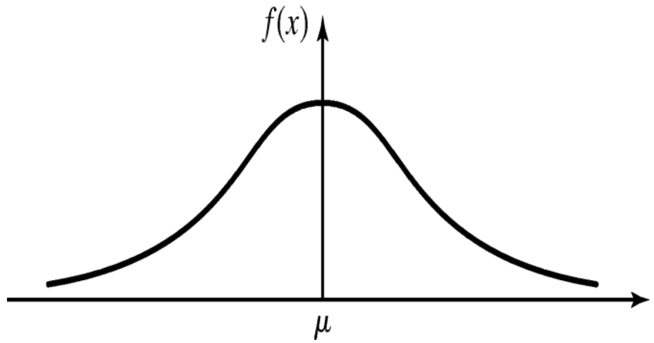
## Identifying the Distribution

Selecting the Family of Distributions

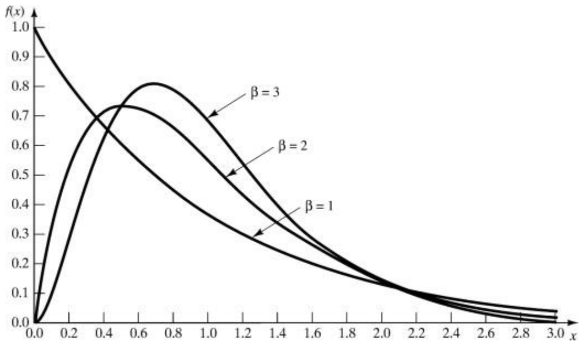
# Selecting the Family of Distributions

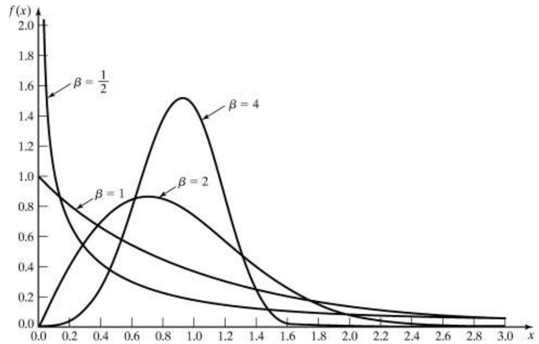
- A family of distributions is selected based on:
  - The context of the input variable
  - Shape of the histogram
- Frequently encountered distributions:
  - Easier to analyze: Exponential, Normal, and Poisson




  - Difficult to analyze: Beta, Gamma, and Weibull



# Selecting the Family of Distributions

- Use the physical basis of the distribution as a guide, e.g.:
  - **Binomial**: Number of successes in n trials
  - **Negative binomial and geometric**: Number of trials to achieve k successes
  - **Poisson**: Number of independent events that occur in a fix amount of time or space
  - **Normal**: Distribution of a process that is the sum of a number of component processes
  - **Lognormal**: Distribution of a process that is the product of a number of component processes
  - **Exponential**: Time between independent events, or a process time that is memoryless
  - **Weibull**: Time to failure for components
  - **Discrete or continuous uniform**: Models complete uncertainty
  - **Triangular**: A process for which only the minimum, most likely, and maximum values are known
  - **Empirical**: Resamples from the actual data collected

# Selecting the Family of Distributions

- Remember the physical characteristics of the process
  - Is the process naturally discrete or continuous valued?

  - Is it bound?

  - Value range?
    - Only positive values
    - Only negative values
    - Interval of [-a:b]

- No “true” distribution for any stochastic input process

- Goal: obtain a good approximation

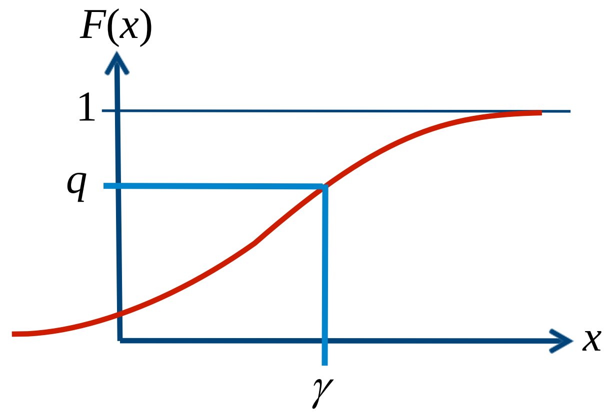
# Identifying the Distribution

Quantile-Quantile Plots

# Quantile-Quantile Plots

- *Q-Q* plot is a useful tool for evaluating distribution fit
- If $X$ is a random variable with CDF $F$, then the $q$-quantile of $X$ is the $\gamma$ such that

$$F(\gamma) = P(X \le \gamma) = q \;\;, \quad \text{for } 0 < q < 1$$




  - When $F$ has an inverse, $\gamma = F^{-1}(q)$

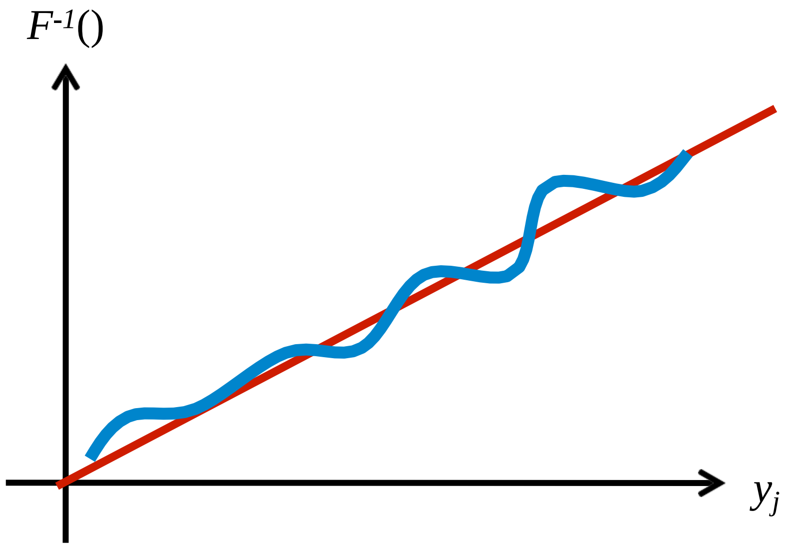
- Let $\{x_i, i = 1,2, \ldots, n\}$ be a sample of data from $X$ and $\{y_j, j = 1,2, \ldots, n\}$ be this sample in ascending order:

$$y_j \text{ is approximately } F^{-1}\left(\frac{j - 0.5}{n}\right)$$

  - where $j$ is the ranking or order number

# Quantile-Quantile Plots

- The plot of $y_j$ versus $F^{-1}((j - 0.5) / n)$ is
  - Approximately a straight line if $F$ is a member of an appropriate family of distributions
  - The line has slope 1 if $F$ is a member of an appropriate family of distributions with appropriate parameter values



# Quantile-Quantile Plots: Example

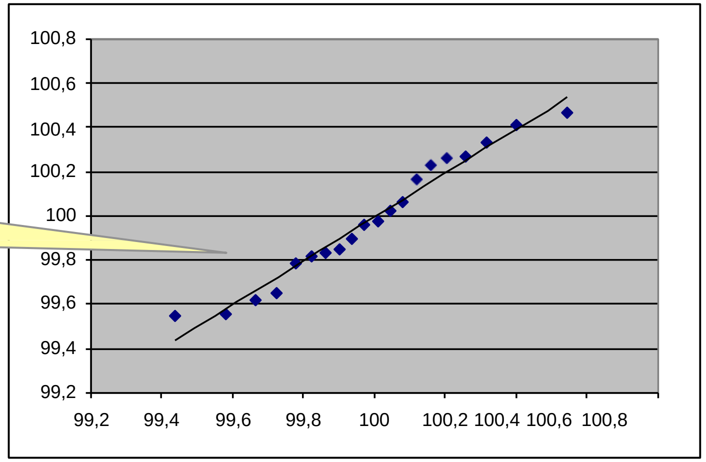
- Example: Door installation times of a robot follows a normal distribution.
  - The observations are ordered from the smallest to the largest
  - $y_j$ are plotted versus $F^{-1}((j - 0.5)/n)$ where $F$ has a normal distribution with the sample mean ($99.99$ sec) and sample variance ($0.2832^2$ sec$^2$)

| j | Value |
|---|---|
| 1 | 99,55 |
| 2 | 99,56 |
| 3 | 99,62 |
| 4 | 99,65 |
| 5 | 99,79 |
| 6 | 99,98 |
| 7 | 100,02 |
| 8 | 100,06 |
| 9 | 100,17 |
| 10 | 100,23 |
| 11 | 100,26 |
| 12 | 100,27 |
| 13 | 100,33 |
| 14 | 100,41 |
| 15 | 100,47 |

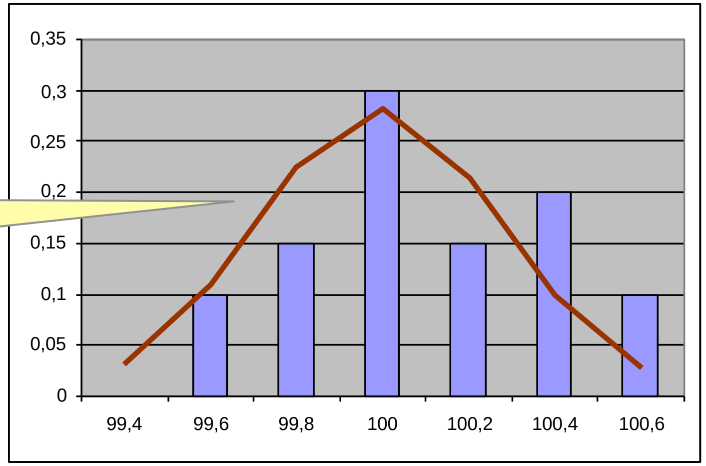
# Quantile-Quantile Plots: Example

- Example (continued): Check whether the door installation times follow a normal distribution.

Straight line, supporting the hypothesis of a normal distribution

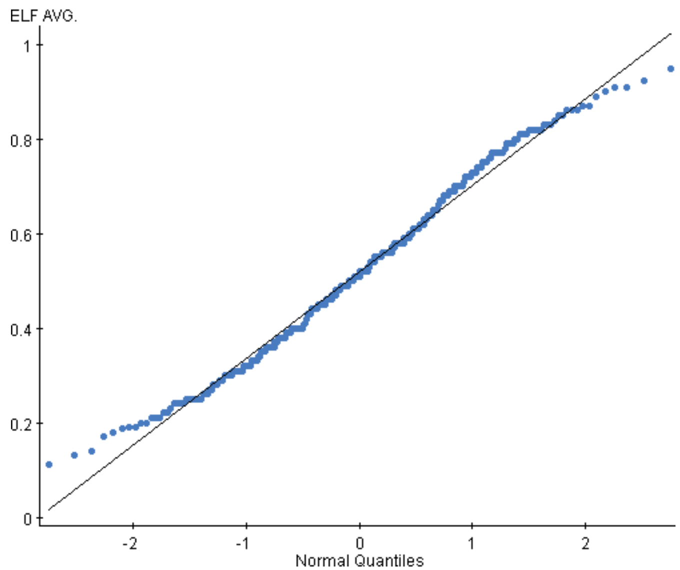


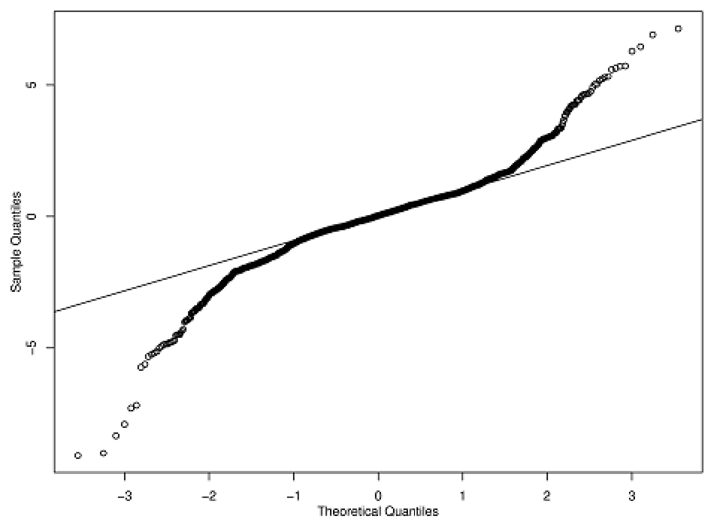

Superimposed density function of the normal distribution



# Quantile-Quantile Plots

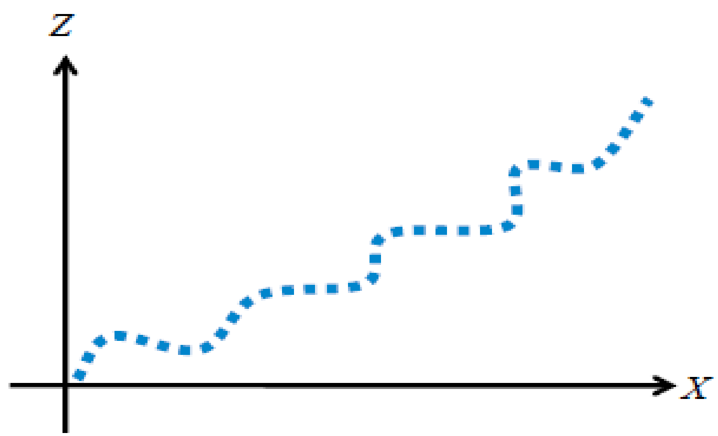
- Consider the following while evaluating the linearity of a Q-Q plot:
  - The observed values never fall exactly on a straight line
  - The ordered values are ranked and hence not independent, unlikely for the points to be scattered about the line
  - Variance of the extremes is higher than the middle. Linearity of the points in the middle of the plot is more important.



# Quantile-Quantile Plots

- *Q-Q* plot can also be used to check homogeneity
  - It can be used to check whether a single distribution can represent two sample sets
  - Given two random variables
    - $X$ and $x_1, x_2, \ldots, x_n$
    - $Z$ and $z_1, z_2, \ldots, z_n$
  - Plotting the **ordered** values of $X$ and $Z$ against each other reveals approximately a straight line if $X$ and $Z$ are well represented by the same distribution



# Parameter Estimation

# Parameter Estimation

- Parameter Estimation: Next step after selecting a family of distributions
- If observations in a sample of size $n$ are $X_1, X_2, \ldots, X_n$ (discrete or continuous), the sample mean and sample variance are:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\left(\sum_{i=1}^{n} X_i^2\right) - n\bar{X}^2}{n-1}$$

- If the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{n} f_j X_j}{n} \qquad S^2 = \frac{\left(\sum_{j=1}^{n} f_j X_j^2\right) - n\bar{X}^2}{n-1}$$

  - where $f_j$ is the observed frequency of value $X_j$

# Parameter Estimation

- When raw data are unavailable (data are grouped into class intervals), the approximate sample mean and variance are:

$$\bar{X} = \frac{\sum_{j=1}^{c} f_j m_j}{n} \qquad S^2 = \frac{\left(\sum_{j=1}^{n} f_j m_j^2\right) - n\bar{X}^2}{n-1}$$

  - $f_j$ is the observed frequency in the $j$-th class interval
  - $m_j$ is the midpoint of the $j$-th interval
  - $c$ is the number of class intervals

- A parameter is an unknown constant, but an estimator is a statistic.

# Parameter Estimation: Example

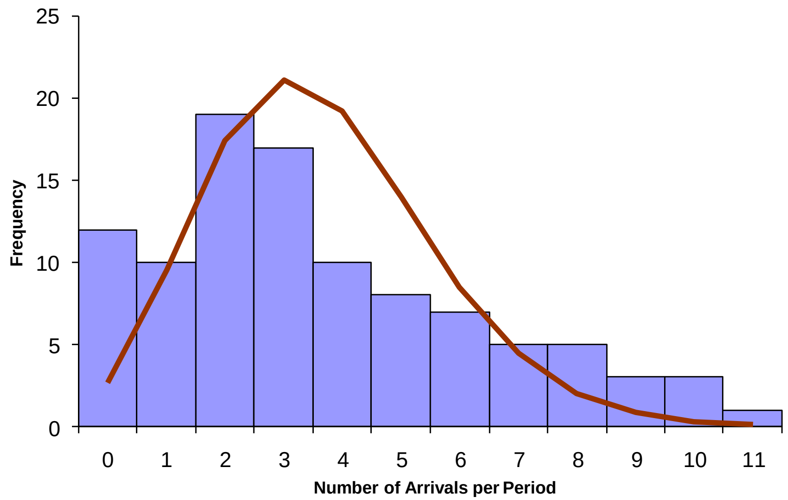
- Vehicle Arrival Example (continued): Table in the histogram of the example on Slide 10 can be analyzed to obtain:

$$n = 100,\ f_1 = 12,\ X_1 = 0,\ f_2 = 10,\ X_2 = 1, \ldots \quad \text{and } \sum_{j=1}^{k} f_j X_j = 364, \text{ and } \sum_{j=1}^{k} f_j X_j^2 = 2080$$

- The sample mean and variance are

$$\bar{X} = \frac{364}{100} = 3.64$$

$$S^2 = \frac{2080 - 100 \cdot (3.64)^2}{99} = 7.63$$




- The histogram suggests $X$ to have a Poisson distribution
  - However, note that sample mean is not equal to sample variance.
  - Theoretically: Poisson with parameter $\lambda \Rightarrow \mu = \sigma^2 = \lambda$
  - Reason: each estimator is a random variable, it is not perfect.

# Parameter Estimation

- Maximum-Likelihood Estimators (MLE)
  - Discrete distribution with one parameter $\theta$ ➡ $p_\theta(x)$
  - Given iid sample $X_1, X_2, \ldots, X_n$
  - Likelihood function $L(\theta)$ is defined as

$$L(\theta) = p_\theta(X_1)\, p_\theta(X_2)\, \ldots\, p_\theta(X_n)$$

  - MLE of the unknown $\theta$ is $\theta'$ given by $\theta$ that maximizes $L(\theta)$ ➡ $L(\theta') \geq L(\theta)$ for all values of $\theta$

# Parameter Estimation

- Maximum-Likelihood Estimators (MLE)
- Suggested estimators for distributions often used in simulation

| Distribution | Parameter | Estimator |
|---|---|---|
| Poisson | $\alpha$ | $\hat{\alpha} = \bar{X}$ |
| Exponential | $\lambda$ | $\hat{\lambda} = \frac{1}{\bar{X}}$ |
| Gamma | $\beta, \theta$ | $\hat{\theta} = \frac{1}{\bar{X}}$ |
| Normal | $\mu, \sigma^2$ | $\hat{\mu} = \bar{X}, \hat{\sigma}^2 = S^2$ |
| Lognormal | $\mu, \sigma^2$ | $\hat{\mu} = \bar{X}, \hat{\sigma}^2 = S^2$ |

After taking *ln* of data.

# Parameter Estimation

- Maximum Likelihood example exponential distribution

# Goodness-of-Fit Tests

# Goodness-of-Fit Tests

- Conduct hypothesis testing on input data distribution using
  - Kolmogorov-Smirnov test
  - Chi-square test

- No single correct distribution in a real application exists
  - If very little data are available, it is unlikely to reject any candidate distributions
  - If a lot of data are available, it is likely to reject all candidate distributions

- Be aware of mistakes in decision finding
  - Type I Error: $\alpha$
    - Error of first kind, False positive
    - Reject $H_0$ when it is true
  - Type II Error: $\beta$
    - Error of second kind, False negative
    - Retain $H_0$ when it is not true

| Statistical Decision | State of the null hypothesis | |
|---|---|---|
| | $H_0$ True | $H_0$ False |
| Accept $H_0$ | Correct | Type II Error<br>Incorrectly accept $H_0$<br>False negative |
| Reject $H_0$ | Type I Error<br>Incorrectly reject $H_0$<br>False positive | Correct |

# Chi-Square Test

- Intuition: comparing the histogram of the data to the shape of the candidate density or mass function
- Valid for large sample sizes when parameters are estimated by maximum-likelihood
- Arrange the $n$ observations into a set of $k$ class intervals
- The test statistic is:

Observed frequency in the $i$-th class

Expected Frequency
$E_i = n \times p_i$
where $p_i$ is the theoretical prob. of the $i$-th interval.
*Suggested Minimum = 5*

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- $\chi_0^2$ approximately follows the Chi-square distribution with $k$-$s$-1 degrees of freedom
  - $s$ = number of parameters of the hypothesized distribution estimated by the sample statistics.

# Chi-Square Test

- The hypothesis of a Chi-square test is
  - $H_0$: The random variable, $X$, conforms to the distributional assumption with the parameter(s) given by the estimate(s).
  - $H_1$: The random variable $X$ does not conform.

$$\text{Test result} \begin{cases} \chi_0^2 \le \chi^2_{\alpha,k-s-1} & \text{Accept } H_0 \\ \chi_0^2 > \chi^2_{\alpha,k-s-1} & \text{Reject } H_0 \end{cases}$$

- If the distribution tested is discrete and combining adjacent cells is not required (so that $E_i >$ minimum requirement):
  - Each value of the random variable should be a class interval, unless combining is necessary, and

$$p_i = p(x_i) = P(X = x_i)$$

# Chi-Square Test

- If the distribution tested is continuous:

$$p_i = \int_{a_{i-1}}^{a_i} f(x)\,dx = F(a_i) - F(a_{i-1})$$

  - where $a_{i\text{-}1}$ and $a_i$ are the endpoints of the $i$-th class interval
  - $f(x)$ is the assumed PDF, $F(x)$ is the assumed CDF
  - Recommended number of class intervals ($k$):

| Sample size (n) | Number of class intervals (k) |
|---:|---|
| 20 | Do not use the chi-square test |
| 50 | 5 to 10 |
| 100 | 10 to 20 |
| > 100 | $\sqrt{n}$ to $\frac{n}{5}$ |

  - Caution: Different grouping of data (i.e., $k$) can affect the hypothesis testing result.

# Chi-Square Test: Example

- Vehicle Arrival Example (continued):

  $H_0$: the random variable is Poisson distributed.

  $H_1$: the random variable is not Poisson distributed.

| $x_i$ | Observed Frequency, $O_i$ | | Expected Frequency, $E_i$ | | $(O_i - E_i)^2/E_i$ |
|---|---|---|---|---|---|
| 0 | 12 | 22 | 2.6 | 12.2 | 7.87 |
| 1 | 10 | | 9.6 | | |
| 2 | 19 | | 17.4 | | 0.15 |
| 3 | 17 | | 21.1 | | 0.8 |
| 4 | 19 | | 19.2 | | 4.41 |
| 5 | 6 | | 14.0 | | 2.57 |
| 6 | 7 | | 8.5 | | 0.26 |
| 7 | 5 | 17 | 4.4 | 7.6 | 11.62 |
| 8 | 5 | | 2.0 | | |
| 9 | 3 | | 0.8 | | |
| 10 | 3 | | 0.3 | | |
| > 11 | 1 | | 0.1 | | |
| | 100 | | 100.0 | | 27.68 |

$$E_i = n \cdot p(x) = n \cdot \frac{e^{-\alpha}\alpha^x}{x!}$$

Combined because of the assumption of min $E_i = 5$, *e.g.*, $E_1 = 2.6 < 5$, *hence combine with* $E_2$

- Degree of freedom is $k$-$s$-1 = 7-1-1 = 5, hence, the hypothesis is rejected at the $\alpha$=0.05 level of significance.

$$\chi_0^2 = 27.68 > \chi^2_{0.05,5} = 11.1$$

# Kolmogorov-Smirnov Test

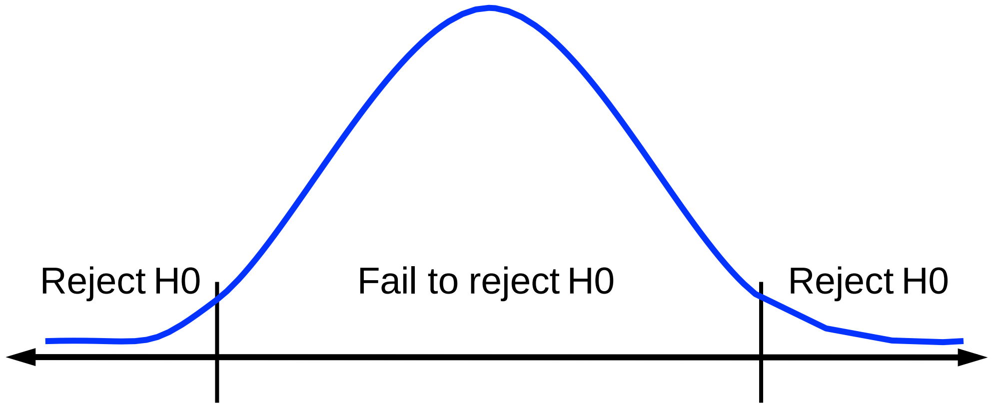
- Intuition: formalize the idea behind examining a Q-Q plot
- Recall
  - The test compares the continuous CDF, $F(x)$, of the hypothesized distribution with the empirical CDF, $SN(x)$, of the $N$ sample observations.
  - Based on the maximum difference statistic:

$$D = max| F(x) - SN(x) |$$

- A more powerful test, particularly useful when:
  - Sample sizes are small
  - No parameters have been estimated from the data
- When parameter estimates have been made:
  - Critical values are biased, too large.
  - More conservative, i.e., smaller Type I error than specified.

# $p$-Values and "Best Fits"

- Hypothesis testing requires a significance level
  - Significance level ($\alpha$) is the probability of falsely rejecting $H_0$
  - Common significance levels
    - $\alpha = 0.1$
    - $\alpha = 0.05$
    - $\alpha = 0.01$
- Be aware that significance level does not tell anything about the subject of the test!
- Generalization of the significance level: $p$-value



# $p$-Values and "Best Fits"

- $p$-*value* for the test statistics
  - The significance level at which one would just reject $H_0$ for the given test statistic value.
  - A measure of fit, the larger the better
  - Large $p$-value: good fit
  - Small $p$-value: poor fit

- Vehicle Arrival Example (cont.):
  - $H_0$: data is Poisson
  - Test statistics: $\chi_0^2 = 27.68$, with 5 degrees of freedom
  - The $p$-*value* $F(5, 27.68) = 0.00004$, meaning we would reject $H_0$ with $0.00004$ significance level, hence Poisson is a poor fit.

## $p$-Values and "Best Fits"

- Many software use $p$-*value* as the ranking measure to automatically determine the "best fit".

- Things to be cautious about:
  - Software may not know about the physical basis of the data, distribution families it suggests may be inappropriate.
  - Close conformance to the data does not always lead to the most appropriate input model.
  - $p$-*value* does not say much about where the lack of fit occurs

- Recommended: always inspect the automatic selection using graphical methods.

# Fitting a Non-stationary Poisson Process

# Fitting a Non-stationary Poisson Process

- Fitting a NSPP to arrival data is difficult, possible approaches:
  - Fit a very flexible model with lots of parameters
  - Approximate constant arrival rate over some basic interval of time, but vary it from time interval to time interval.
- Suppose we need to model arrivals over time $[0, T]$, our approach is the most appropriate when we can:
  - Observe the time period repeatedly
  - Count arrivals / record arrival times
  - Divide the time period into $k$ equal intervals of length $\otimes t = T/k$
  - Over $n$ periods of observation let $C_{ij}$ be the number of arrivals during the $i$-th interval on the $j$-th period

# Fitting a Non-stationary Poisson Process

- The estimated arrival rate during the $i$-th time period $(i\text{-}1)\,\Delta t < t \le i\,\Delta t$ is:

$$\hat{\lambda}(t) = \frac{1}{n\Delta t}\sum_{j=1}^{n} C_{ij}$$

  - $n$ = Number of observation periods
  - $\Delta t$ = Time interval length
  - $C_{ij}$ = Number of arrivals during the $i$-th time interval on the $j$-th observation period
- Example: Divide a 10-hour business day [8am,6pm] into equal intervals $k = 20$ whose length $\Delta t = ½$, and observe over $n=3$ days

| Time Period | Number of Arrivals | | | Estimated Arrival Rate (arrivals/hr) |
|---|---|---|---|---|
| | Day 1 | Day 2 | Day 3 | |
| 8:00 - 8:30 | 12 | 14 | 10 | 24 |
| 8:30 - 9:00 | 23 | 26 | 32 | 54 |
| 9:00 - 9:30 | 27 | 18 | 32 | 52 |
| 9:30 - 10:00 | 20 | 13 | 12 | 30 |

For instance,
1/3(0.5)*(23+26+32)
= 54 arrivals/hour

# Selecting Models without Data

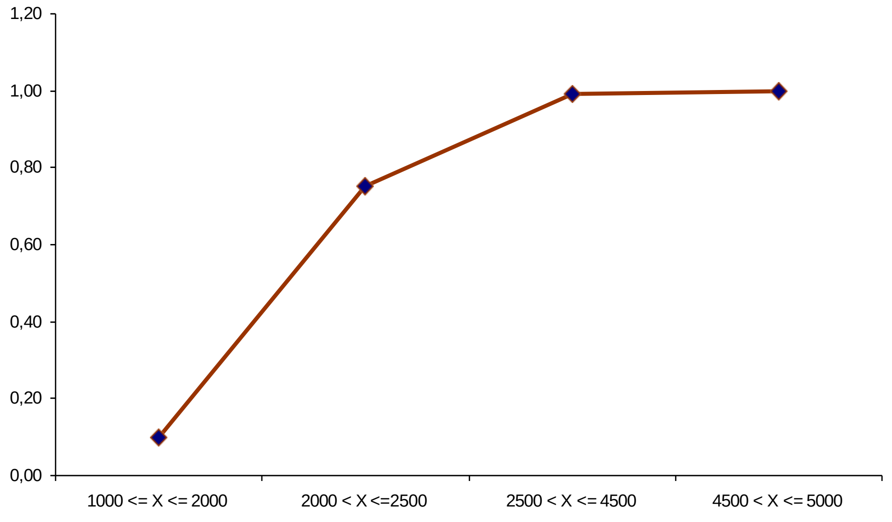
# Selecting Models without Data

- If data is not available, some possible sources to obtain information about the process are:
  - **Engineering data:** often product or process has performance ratings provided by the manufacturer or company rules specify time or production standards.
  - **Expert option:** people who are experienced with the process or similar processes, often, they can provide optimistic, pessimistic and most-likely times, and they may know the variability as well.
  - **Physical or conventional limitations:** physical limits on performance, limits or bounds that narrow the range of the input process.
  - **The nature of the process**.
- The uniform, triangular, and beta distributions are often used as input models.
  - Speed of a vehicle?

# Selecting Models without Data

- Example: Production planning simulation.
  - Input of sales volume of various products is required, salesperson of product XYZ says that:
    - No fewer than 1000 units and no more than 5000 units will be sold.
    - Given her experience, she believes there is a 90% chance of selling more than 2000 units, a 25% chance of selling more than 2500 units, and only a 1% chance of selling more than 4500 units.
  - Translating these information into a cumulative probability of being less than or equal to those goals for simulation input:

| $i$ | Interval (Sales) | PDF | Cumulative Frequency, $c_i$ |
|---|---|---|---|
| 1 | $1000 \leq X \leq 2000$ | 0.1 | 0.10 |
| 2 | $2000 < X \leq 2500$ | 0.65 | 0.75 |
| 3 | $2500 < X \leq 4500$ | 0.24 | 0.99 |
| 4 | $4500 < X \leq 5000$ | 0.01 | 1.00 |



# Multivariate and Time-Series Input Models

# Multivariate and Time-Series Input Models

- The random variable discussed until now were considered to be independent of any other variables within the context of the problem
  - However, variables may be related
  - If they appear as input, the relationship should be investigated and taken into consideration
- Multivariate input models
  - Fixed, finite number of random variables $X_1, X_2, \ldots, X_k$
  - For example, lead time and annual demand for an inventory model
  - An increase in demand results in lead time increase, hence variables are dependent.
- Time-series input models
  - Infinite sequence of random variables, e.g., $X_1, X_2, X_3, \ldots$
  - For example, time between arrivals of orders to buy and sell stocks
  - Buy and sell orders tend to arrive in bursts, hence, times between arrivals are dependent.

# Covariance and Correlation

- Consider a model that describes relationship between $X_1$ and $X_2$:

$$(X_1 - \mu_1) = \beta(X_2 - \mu_2) + \varepsilon$$

ε is a random variable with mean 0 and is independent of $X_2$

  - $\beta = 0$, $X_1$ and $X_2$ are statistically independent
  - $\beta > 0$, $X_1$ and $X_2$ tend to be above or below their means together
  - $\beta < 0$, $X_1$ and $X_2$ tend to be on opposite sides of their means

- Covariance between $x_1$ and $x_2$:

$$\mathrm{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E(X_1 X_2) - \mu_1 \mu_2$$

- Covariance between $X_1$ and $X_2$:

  - where $\mathrm{cov}(X_1, X_2)\begin{cases} = 0 \\ < 0 \\ > 0 \end{cases} \Rightarrow \beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$ $\qquad -\infty < \mathrm{cov}(X_1, X_2) < \infty$

# Covariance and Correlation

- Correlation between $X_1$ and $X_2$ (values between -1 and 1):

$$\rho = \operatorname{corr}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

- where $$corr(X_1, X_2)\begin{cases} = 0 \\ < 0 \\ > 0 \end{cases} \Rightarrow \quad \beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$$

- The closer $\rho$ is to -1 or 1, the stronger the linear relationship is between $X_1$ and $X_2$.

# Time-Series

- A time series is a sequence of random variables $X_1, X_2, X_3, \ldots$ which are identically distributed (same mean and variance) but dependent.
  - $cov(X_t, X_{t+h})$ is the lag-$h$ autocovariance
  - $corr(X_t, X_{t+h})$ is the lag-$h$ autocorrelation
  - If the autocovariance value depends only on $h$ and not on $t$, the time series is **covariance stationary**
  - For covariance stationary time series, the shorthand for lag-$h$ is used

$$\rho_h = corr(X_t, X_{t+h})$$

- Notice
  - autocorrelation measures the dependence between random variables that are separated by $h$-1 others in the time series

# Multivariate Input Models

- If $X_1$ and $X_2$ are normally distributed, dependence between them can be modeled by the bivariate normal distribution with $\mu_1$, $\mu_2$, $\sigma_1^2$, $\sigma_2^2$ and correlation $\rho$
  - To estimate $\mu_1$, $\mu_2$, $\sigma_1^2$, $\sigma_2^2$, see "Parameter Estimation"
  - To estimate $\rho$, suppose we have $n$ independent and identically distributed pairs $(X_{11}, X_{21})$, $(X_{12}, X_{22})$, … $(X_{1n}, X_{2n})$,
  - Then the sample covariance is

$$\hat{\text{cov}}(X_1, X_2) = \frac{1}{n-1} \sum_{j=1}^{n} (X^{1j} - \bar{X}_1)(X_{2j} - \bar{X}_2)$$

  - The sample correlation is

$$\hat{\rho} = \frac{\hat{\text{cov}}(X_1, X_2)}{\hat{\sigma}_1 \hat{\sigma}_2}$$

Sample deviation

# Multivariate Input Models: Example

- Let $X_1$ the average lead time to deliver and $X_2$ the annual demand for a product.
- Data for 10 years is available.

$$\bar{X}_1 = 6.14, \quad \bar{\sigma}_1 = 1.02$$

$$\bar{X}_2 = 101.8, \quad \bar{\sigma}_2 = 9.93$$

$$\hat{\text{cov}}_{\text{sample}} = 8.66$$

Covariance

$$\hat{\rho} = \frac{8.66}{1.02 \times 9.93} = 0.86$$

| Lead Time ($X_1$) | Demand ($X_2$) |
|---|---|
| 6,5 | 103 |
| 4,3 | 83 |
| 6,9 | 116 |
| 6,0 | 97 |
| 6,9 | 112 |
| 6,9 | 104 |
| 5,8 | 106 |
| 7,3 | 109 |
| 4,5 | 92 |
| 6,3 | 96 |

- Lead time and demand are strongly dependent.
  - Before accepting this model, lead time and demand should be checked individually to see whether they are represented well by normal distribution.

# Time-Series Input Models

- If $X_1, X_2, X_3, \ldots$ is a sequence of identically distributed, but dependent and covariance-stationary random variables, then we can represent the process as follows:
  - Autoregressive order-1 model, **AR(1)**
  - Exponential autoregressive order-1 model, **EAR(1)**

- Both have the characteristics that:

$$\rho_h = corr(X_t, X_{t+h}) = \rho^h, \quad \text{for } h = 1,2,\ldots$$

  - Lag-$h$ autocorrelation decreases geometrically as the lag increases, hence, observations far apart in time are nearly independent

# Time-Series Input Models: Autoregressive order-1 model AR(1)

- Consider the time-series model:

$$X_t = \mu + \phi(X_{t-1} - \mu) + \varepsilon_t, \qquad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. normally distributed with $\mu_\varepsilon = 0$ and variance $\sigma_\varepsilon^2$

- If initial value $X_1$ is chosen appropriately, then
  - $X_1, X_2, \ldots$ are normally distributed with *mean* = $\mu$, and *variance* = $\sigma^2/(1-\phi^2)$
  - Autocorrelation $\rho_h = \phi^h$

- To estimate $\phi, \mu, \sigma_\varepsilon^2$:

$$\hat{\mu} = \bar{X}, \qquad \hat{\sigma}_\varepsilon^2 = \hat{\sigma}^2(1-\hat{\phi}^2), \qquad \hat{\phi} = \frac{\hat{\mathrm{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\mathrm{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

# Time-Series Input Models: Exponential AR(1) model EAR(1)

- Consider the time-series model:

$$X_t = \begin{cases} \phi X_{t-1}, & \text{with probability } \phi \\ \phi X_{t-1} + \varepsilon_t, & \text{with probability } 1\text{-}\phi \end{cases} \qquad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. exponentially distributed with $\mu_\varepsilon = 1/\lambda$, and $0 \le \phi < 1$

- If $X_1$ is chosen appropriately, then
  - $X_1, X_2, \ldots$ are exponentially distributed with *mean* $= 1/\lambda$
  - Autocorrelation $\rho_h = \phi^h$, and only positive correlation is allowed.
- To estimate $\phi, \lambda$ :

$$\hat{\lambda} = \frac{1}{\bar{\bar{X}}}, \qquad \hat{\phi} = \hat{\rho} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

# Summary

- In this chapter, we described the 4 steps in developing input data models:
  (1) Collecting the raw data
  (2) Identifying the underlying statistical distribution
  (3) Estimating the parameters
  (4) Testing for goodness of fit