The **Acceptance-Rejection Method** is a widely used technique for generating random variate from a specified probability distribution, especially when the direct sampling methods (like the Inverse-Transform Method) are not feasible. This method is particularly useful for distributions that do not have a straightforward inverse CDF or for complex distributions.

General Steps of the Acceptance-Rejection Method

1. **Define the Target Distribution**:
   - Identify the probability distribution f(x) from which you want to generate samples. This is often the distribution of interest.
2. **Choose a Proposal Distribution**:
   - Select a simpler distribution g(x) from which random samples can be easily generated. This proposal distribution should ideally be similar to the target distribution.
3. **Determine the Scaling Constant M**:
   - Find a constant M such that:

$$M \cdot g(x) \geq f(x) \quad \text{for all } x$$

   This ensures that the scaled proposal distribution covers the target distribution.

4. **Generate Random Samples**:

- Follow these steps until a valid sample is accepted:
  1. **Sample X**: Generate a random variate X from the proposal distribution g(x).
  2. **Sample U**: Generate a uniform random number U from U(0,1).
  3. **Acceptance Criterion**: Accept X if:

$$U \leq \frac{f(X)}{M \cdot g(X)}$$

  If this condition fails, reject X and return to step 1.

Key Concepts

1. **Independence**: The method assumes that the random samples generated from g(x) and U is independent.
2. **Efficiency**: The efficiency of the Acceptance-Rejection Method depends on how well the proposal distribution approximates the target distribution. A good choice of g(x) leads to a higher acceptance rate.
3. **Range of x**: It's essential to consider the support of the target distribution when choosing g(x). The proposal distribution should cover the range where the target distribution is significant.

Example

Problem Setup

Suppose you want to generate random variates from a target distribution f(x), which could be a complex or less well-known distribution, such as a custom probability distribution with a complicated shape.

1. **Define Target and Proposal Distributions**:
   - **Target**: f(x) (e.g., a complicated distribution).
   - **Proposal**: g(x) (e.g., a normal or uniform distribution).
2. **Determine MMM**:
   - Calculate MMM such that the condition $M \cdot g(x) \geq f(x)$ holds for all relevant values of x.
3. **Generate Samples**:
   - Use the steps outlined above to generate samples from f(x) by repeatedly sampling from g(x) and applying the acceptance criterion.

Applications

- **Flexible Sampling**: The method is applicable to a wide range of distributions, making it a versatile choice in many simulation scenarios.
- **Complex Distributions**: It allows sampling from distributions that may not have an explicit form for the inverse CDF.
- **Statistical Analysis**: Commonly used in Monte Carlo simulations, Bayesian statistics, and other areas of research where random sampling is required.

Example Calculation

Let's use a Poisson distribution with $\lambda = 4$ and a geometric distribution as the proposal:

1. **Target PMF** (Poisson):

$$P(X = k) = \frac{4^k e^{-4}}{k!}$$

2. **Proposal PMF** (Geometric, with $p = 0.25$):

$$g(k) = 0.25(0.75)^k$$

3. **Calculate the Ratio for Different k:**

- For $k = 0$:

$$P(X = 0) = e^{-4} \approx 0.0183$$

$$g(0) = 0.25 \quad \Rightarrow \quad R(0) = \frac{0.0183}{0.25} \approx 0.0732$$

- For $k = 1$:

$$P(X = 1) = 4e^{-4} \approx 0.0732$$

$$g(1) = 0.25(0.75) \approx 0.1875 \quad \Rightarrow \quad R(1) = \frac{0.0732}{0.1875} \approx 0.3904$$

- For $k = 2$:

$$P(X = 2) = \frac{16e^{-4}}{2} \approx 0.0735$$

$$g(2) = 0.25(0.75)^2 \approx 0.140625 \quad \Rightarrow \quad R(2) = \frac{0.0735}{0.140625} \approx 0.522$$

- Continue this for higher values of k until the ratio stabilizes.

4. **Find Maximum $R(k)$:**

- Let's say after evaluating several k, you find that M is the maximum ratio from all calculated values, e.g., $M \approx 0.522$.