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Random Variate Generation

→ In modeling and Simulation random numbers from a variety of probability distribⁿ functions are generated to simulate the behaviour of random events occuring in the future.

→ RVG is used to simulate the uncertainity in behaviour of an entity in the system.

→ If the form of randomness of the entity is known the entity is said to follow a particular probability distribution, else randomness is simulated by empirical distribution.

→ In efficient generⁿ of these nos can be a significant bottleneck for simulation applications. Generating these random numbers imprecisely can skew results.

→ once it has been established that a good uniform (0,1) random number generator is available then the other statistical distributions may be generated from stream of uniform (0,1) pseudo-random numbers.

→ A random variate is a variable generated from uniformly distributed pseudo random numbers.

→ A Random Variate also refers to a particular value of random variable. A random observations of a random variable that form a desired probability distribⁿ are called random variats. and the algorithms that generates random variants for a particular probability distribution is called Random variate generator.

→ For example a random variate generator for normal distribution generates random variants that satisfy normal probability distribution.

→ Random variate generators are of two types:- univariate and multi variate. A univariate RVG involves the generation of single variate at a time, multi variate involves generation of vector of variates at a time, which do not demonstrate mutual independence.

*→ Statistical and simulation applications require the inherent random phenomena to be characterized before running the model. A random phenomenon is characterized by collecting observations, fitting an appropriate probability distribution to the observations and estimation of the parameters of fitted probability distributions. Once the random phenomenon has been characterized, the model calls the appropriate random variate generator with the estimated values of parameters. The generated random variates are then used to run the model. The random variate generation process is given in fig below.

Uniform Random Number Generation → Convert to Desired Distribution → Random Nos with Specified Distributions

Random variate generation Process

* → There are various techniques available for random variate genern for common probability distribution. Some probability distribution have alternative algorithms also,

* Some widely used techniques for generating random variants are
  1) Inverse Transformation.
  2) Convolution.
  3) Acceptance - Rejection.

Inverse Transformation Method.

→ Let X be a continuously distributed random variable. Then its probability density function is given by

$$P(a < x < b) = \int_a^b f(x)\,dx.$$

→ The distribution function can be obtained from density function, which is.

$$F(x) = \int_{-\infty}^{\infty} f(t)\,dt.$$

→ The distribution function being monotonically increasing and continuous the eqn $F(x) = R$ (for any $0 < R < 1$) can be solved to give a unique X or written alternatively, $X = F^{-1}(R)$. This essentially is the inverse x formation method for generating random variants.

→ It can be used to generate random variate from exponential uniform weibull triangular and empirical; discrete distributions.

→ The Probability integeral xformation theorem states that if R_i $(i = 1, 2, 3 ---)$ are independent, uniformly distributed random numbers over the period 0 to 1 and $F^{-1}(X)$ is the inverse of cumulative distribution function for random variable X then the random variables defined by $X_i = F^{-1}(R_i)$ are a random sample of variable X.

→ In other words, we can say that to produce random nos with a given distribution the inverse cumulative distribut function must be evaluated with a sequence of uniformly distributed nos. in the range 0 to 1.

→ This technique is most useful when the CDF $F(x)$ has simple form so that its inverse can be easily computed.

→ The generalized algorithm for inverse xform technique is given below and can be made specialized as per the distribution.

Algorithm:-

1) Get the CDF of desired random variable X.
2) Generate $R \sim U(0,1)$ and set $F(X) = R$ on the range of X.
3) Calculate F^{-1} by solving the equation $F(X) = R$ in terms of R which returns $X = F^{-1}(R)$.
4) Generate uniform random nos $R_1, R_2 ---$ from $(0,1)$ and use $X_i = F^{-1}(R_i)$, to generate variants.

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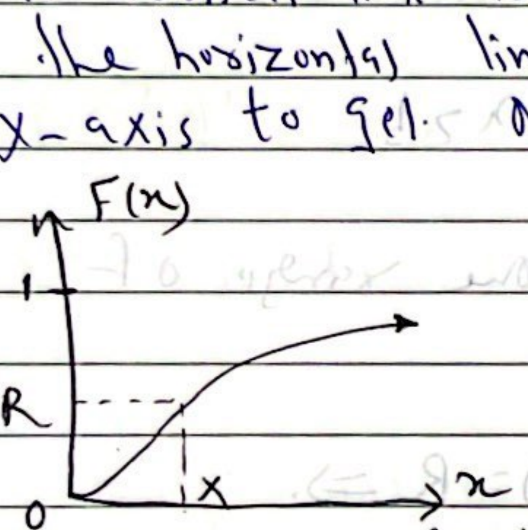
Graphical Method :-

1) Plot the CDF, $F(x)$.
2) For generating a random variate x with CDF, $F(x)$, first generate a random number R between 0 and 1.
3) Draw a horizontal line through vertical axis at $F(x) = R$ so that it intersects the graph of CDF.
4) A vertical line is droped from point where the horizontal line bisects the CDF on the x-axis to get the random variate X.



Graphical view of inverse xform^n Meth

Inverse Transformation Method for Continuous Distrib

1) Exponential

Step 1) Let the CDF of desired R.V x -

$F(x) = 1 - e^{-\lambda x}, x \geq 0$.

2) Set $F(x) = R$ on the range of $x \Rightarrow 1 - e^{-\lambda x} = R$.

3) Solve the eqn $F(X) = R$.

$$1 - e^{-\lambda x} = R$$
$$e^{-\lambda x} = 1 - R$$
$$-\lambda x = \ln(1-R).$$
$$X = -\frac{1}{\lambda} \ln(1-R).$$

4) Generate uniform random nos $R_1, R_2 \cdots$ from $(0,1)$ and calculate desired random variate by

$$X_i = -\frac{1}{\lambda} \ln(1-R_i).$$

2) Uniform Distribution

Step 1) Get the CDF of desired R.V. X.

$$F(x) = 0 \quad ; x < a.$$
$$= \frac{x-a}{b-a} \quad ; a \le x \le b.$$
$$= 1 \quad ; x > b.$$

2) Set $F(x) = R$ on the range of $X \Rightarrow \frac{X-a}{b-a} = R$.

3) Solve the eqn $F(X) = R \Rightarrow$.

$$\frac{X-a}{b-a} = R.$$
$$X - a = (b-a)R$$
$$X = a + (b-a)R.$$

4) Generate uniform random nos $R_1, R_2 \cdots$ from $(0,1)$ and calculate desired random variable by

$$X_i = a + (b-a)R_i$$

3) Weibull Distribution

Step 1) :- Let the CDF of desired R.V X -

$F(x) = 0 , \quad x < 0.$

$= 1 - e^{-(\frac{x}{\alpha})^{\beta}} \quad x \geq 0.$

2) Set $F(x) = R$ on the range of $X \Rightarrow 1 - e^{-(x/\alpha)^{\beta}} = R.$

3) Solve the eqn $F(x) = R. \Longrightarrow$

$1 - e^{-(x/\alpha)^{\beta}} = R.$

$e^{-(\frac{x}{\alpha})^{\beta}} = 1 - R.$

$-(\frac{x}{\alpha})^{\beta} = \ln(1-R).$

$(x/\alpha)^{\beta} = -\ln(1-R).$

$X = \alpha[-\ln(1-R)]^{1/\beta}.$

4) Generate uniform random nos $R_1, R_2 \ldots$ from (0,1) and calculate desired random Variate by

$X_i = \alpha[-\ln(1-R_i)]^{1/\beta}.$

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Example :- A system has a component whose time to failure is exponentially distributed with failure rate 1/6. Generate two random failure times from this distribution. Use $R_1 = 0.38$ and $R_2 = 0.45$

⇒ The exponential random variate is.

$$X_i = -\frac{1}{\lambda} \ln(1-R_i)$$

Given that failure rate, $\lambda = 1/6$.

$$X_i = -1/1/6 \ln(1-R_i).$$
$$= -6 \ln(1-R_i).$$

Now $X_1 = -6 \ln(1-R_i)$
$= -6 \ln(1-0.38) = \underline{2.868}$

and $X_2 = -6 \ln(1-R_2)$
$= -6 \ln(1-0.45) = \underline{3.587}$

Example :- The time required for Professor to travel from his house to college is uniformly distributed over the interval 30 to 35 minutes. Generate two random travel times from this distribution. Use $R_1 = 0.5023$ and $R_2 = 0.2916$.

⇒ The uniform random variate is

$$X_i = a + (b-a) R_i$$

Given that $a = 30$ and $b = 35$

$$X_i = 30 + (35-30) R_i$$
$$= 30 + 5 R_i$$

∴ $X_1 = 30 + 5R_1 = 30 + 5(0.5023) = \underline{32.5115}$

and $X_2 = 30 + 5R_2 = 30 + 5(0.2916) = \underline{31.458}$.

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Example:- The life time of a computer chip measure in hours in Weibull distributed with parameter $\alpha = 0.2$, $\beta = 0.5$ and $\nu = 0$. Generate two random life time of computer chip from this distribution? Use $R_1 = 0.6173$ and $R_2 = 0.4829$.

$\Rightarrow$ The Weibull variate is:-

$$X_i = \alpha\left[-\ln(1-R_i)\right]^{1/\beta}$$

Given that $\alpha = 0.2$ and $\beta = 0.5$

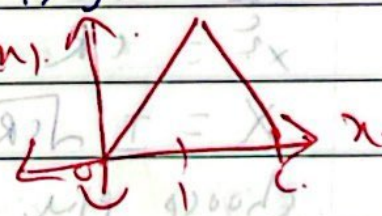
$$X_i = 0.2\left[-\ln(1-R_i)\right]^{1/0.5}$$

$$x_1 = 0.2\left[-\ln(1-0.6173)\right]^{2}$$

$$= \underline{0.1845}$$

and
$$x_2 = 0.2\left[-\ln(1-0.4829)\right]^{2}$$

$$= \underline{0.08699}$$



Triangular Distribution

$\rightarrow$ If we have $X \sim \text{triang}[0, 1, (c-a)/(b-a)]$,

then $x' = a + (b-a)x \sim \text{triang}(a, b, c)$

$\rightarrow$ Hence we restrict our discussion to triang $(0, 1, c)$ random variables so that distribution function is easily inverted to get triangular variate where $0 < c < 1$.

$\rightarrow$ The pdf of triang $(0, 1, c)$ is given by

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$$f(x) = \begin{cases} \dfrac{2(x-0)}{(1-0)(c-0)} = \dfrac{2x}{c} ; & 0 \le x \le 1. \\[2ex] \dfrac{2(c-x)}{(c-1)(c-0)} = \dfrac{2(c-x)}{c(c-1)} ; & 1 \le x \le c \end{cases}$$

→ Algorithm

1) Get the CDF of desired R.V. X using its pdf,

$$F(x) = \begin{cases} 0 & x \le 0. \\[1ex] \dfrac{(x-0)^2}{(1-0)(c-0)} = \dfrac{x^2}{c} ; & 0 \le x \le 1. \\[2ex] 1 - \dfrac{(c-x)^2}{(c-1)(c-0)} = 1 - \dfrac{(c-x)^2}{c(c-1)} ; & 1 \le x \le c. \\[2ex] 1 ; & x > c. \end{cases}$$

2) Set $F(x) = R$ on the range of $X \Rightarrow \dfrac{x^2}{c} = R$ for $0 \le X \le 1$ and $1 - \dfrac{(c-X)^2}{c(c-1)} = R$ for $1 \le X \le c$

3) Solve the eqn $F(X) = R \Rightarrow$

For $0 \le X \le 1$.

$\dfrac{X^2}{c} = R$

$X^2 = cR$

$X = \pm \sqrt{cR}$

choose plus as F^{-1} should not be -ve.

For $1 \le X \le c$.

$1 - \dfrac{(c-X)^2}{c(c-1)} = R$

$\dfrac{(c-X)^2}{c(c-1)} = 1 - R$

$(c-X)^2 = c(c-1)(1-R)$

$c - X = \pm \sqrt{c(c-1)(1-R)}$

$X = c \pm \sqrt{c(c-1)(1-R)}$, choose minus as $F^{-1}(0.5) = 1$.

4) Generate uniform random nos R_1, R_2 --- from [0,1] and calculate desired random variate by

$X_i = \sqrt{cR_i}$ if $0 < R_i \leq 1/c$.

Else $X_i = c - \sqrt{c(c-1)(1-R_i)}$ if $\frac{1}{c} < R_i \leq 1$.

Example :- A digital sensor is used to determine the quality of computer chips. The quality control departement rejects chips those fails which is triangularly distributed with $a=0$, $b=1$ and $c=3$ Generate two random rejected samples from this distrib. Use $R_1 = 0.9536$. and $R_2 = 0.2941$

⟹ Solution :-

The triangular variate is

$X_i = \sqrt{cR_i}$ for $0 < R_i \leq \frac{1}{c}$.

and $X_i = c - \sqrt{(c-1)(1-R_i)}$ for $\frac{1}{c} < R_i \leq 1$

Given that $c = 3$.

$X_i = \sqrt{3R_i}$ for $0 < R_i \leq \frac{1}{3}$.

and $X_i = 3 - \sqrt{3(3-1)(1-R_i)}$ for $\frac{1}{3} < R_i \leq 1$

$= 3 - \sqrt{6(1-R_i)}$

Now $R_1 = 0.9536$ but $\frac{1}{3} < R_1 = 0.9536 \leq 1$.

$X_1 = 3 - \sqrt{6(1-R_1)}$

$= 3 - \sqrt{6(1-0.9536)} = 2.472$

and $R_2 = 0.2941$, but $0 \leq R_2 = 0.2941 \leq \frac{1}{3}$.

$X_2 = \sqrt{3R_2} = \sqrt{3(0.2941)} = 0.939$

Empirical Continuous Distributions

1) It is used when no known theoretical distributions can be found to be a good ~~model~~ model for the input data.

2) We resample the original individual observation itself when input process is known to take on a finite no values.

3) If the data are drawn from continuous distribn then interpolate between the observed data points to fill in the gaps.

4) In case of original data as well as grouped data we used table - look up method for generating random variate from empirical continuous distribution.

5) A freqn distribution table is generated showing the interval and its corresponding probability, cumulative probability and slope.

6) In case of continuous data, since all values betwn tabulated values are possible, so an interpolation betwn tabulated values is reqd.

7) If generated random no falls in an interval betwn two tabulated observations the desired random variate is taken to be the o/p value at lower tabulated observation plus an increment that divides the o/p interval in same proportion that the input divides the input interval.

8) This is consistent with the assumption that the cumulative distribn is approximated by straight line segments betwn the tabulated points.

9) The slopes of these line segments are required to carry out interpolan process numerically these slopes can be calculated and stored in table for generan of random nos in future.

10) Since we are using inverse of distribn function the slope being calculated is the slope of x w.r.t y.

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11) Formally in table lookup method, the input random number 'R' is compared with cumulative freqn. If it is equal to one of the value of cumulative freqn then, the corresponding o/p value is taken.

12) The algorithm for computation of empirical continuous random variated from original and group data is given below.

Algorithm (original data):-

1) Sort the data points in increasing order.
2) Generate a freqn distribn table and assign a probability to each interval as 1/n, where n is the no. of observations.

i	Interval $(x_{i-1} < x \le x_i)$	Probability $1/n$	Cumulative Probability i/n	Slope a_i
1				
2				
:				
n				

3) Draw observed empirical CDF, $\hat{F}(x)$ and follow the graphical method.

4) compute the slope 'a_i' of ith line segment.

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{i/n - (i-1)/n} = \frac{x_i - x_{(i-1)}}{1/n}$$

$$= n\,(x_{(i)} - x_{(i-1)})$$

5) Let the CDF of desired R.V. x -

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$$F(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ \frac{i-1}{n} + \frac{x - X_{(i-1)}}{n(X_{(i)} - X_{(i-1)})} & x_{i-1} \le x \le x \\ 1 & \text{if } x_n \le x \end{cases}$$

6) Set $F(x) = R$ on the range of X:

$$\frac{i-1}{n} + \frac{X - x_{i-1}}{n(x_i - x_{i-1})} = R.$$

7) Compute the inverse of empirical CDF by solving the eqn $\hat{F}(x) = R \Longrightarrow$

$$\frac{i-1}{n} + \frac{\cancel{X - x_{i-1}}}{\cancel{n(x_i - x_{i-1})}} \frac{X - x_{i-1}}{n(x_i - x_{i-1})} = R$$

$$\frac{X - x_{i-1}}{n(x_i - x_{i-1})} = R - \frac{i-1}{n}$$

$$X - x_{i-1} = n(x_i - x_{i-1})\left(R - \frac{i-1}{n}\right)$$

$$X = x_{i-1} + n(x_i - x_{i-1})\left(R - \frac{i-1}{n}\right)$$

$$X = x_{i-1} + a_i\left(R - \frac{i-1}{n}\right)$$

8) Generate uniform random numbers $R_1, R_2 \dots$ and compute desired random variable by

$$X_j = \hat{F}^{-1}(R_j) = x_{i-1} + a_i\left(R_j - \frac{i-1}{n}\right)$$

if $(i-1)/n < R_j \le i/n$.

Example :- Five observations of Fire Brigade station response times to incoming alarms have been collected and are used in a simulation to investigate alternate staffing and fire brigade van scheduling policies. The response times are – 1.60, 0.55, 1.12, 2.20, 1.94. Set-up a table for generating response time by table lookup method and generate two value of response times using uniform random nos 0.5426 and 0.1524.

⇒ Solution 1) Sort the data points in increasing order and let it be $x_{(1)} \le x_{(2)} \dots \le x_{(n)}$

∴ 0.55 1.12 1.60 1.94 2.20.

Since smallest value is 0.55, so we define here $x_{(0)} = 0$.

2) Generate freq distribution table and assign a probability 1/n to each interval.

Given that n = 5 observations.

∴ 1/5 is the probability of each interval.

i	interval $(x_{i-1}) < x \le x_i$	Probability $1/n$	Cumulative Probability, i/n	slope $a_i = n(x_{(i)} - x_{(i-1)})$
1	$0.0 < x \le 0.55$	0.2	0.2	2.75
2	$0.55 < x \le 1.12$	0.2	0.4	2.85
3	$1.12 < x \le 1.60$	0.2	0.6	2.4
4	$1.60 < x \le 1.94$	0.2	0.8	1.7
5	$1.94 < x \le 2.20$	0.2	1.0	1.3

→ Summary of response time data compute the a_i slope.

$a_1 = 5(x_1 - x_0) = 5(0.55 - 0) = 2.75$

$a_2 = 5(x_2 - x_1) = 5(1.12 - 0.55) = 2.85$

$a_3 = 5(x_3 - x_1) = 5(1.60 - 1.12) = 2.4$

$a_4 = 5(x_4 - x_3) = 5(1.94 - 1.60) = 1.7$

$a_5 = 5(x_5 - x_4) = 5(2.20 - 1.94) = 1.3$

The empirical random variate of original data (individual observation) is

$$X_j = \hat{F}^{-1}(R_j) = x_{i-1} + a_i\left(R_j - \frac{i-1}{n}\right)$$

when $\frac{(i-1)}{n} < R_j \le i/n$.

Given that

$R_1 = 0.5426$ and $R_2 = 0.1524$

Now, $R_1 = 0.5426$ which is lying betwn $2/5 = 0.4$ and $3/5 = 0.6$ i.e third interval, hence $i = 3$

$$\Rightarrow X_1 = x_{3-1} + a_3\left(R_1 - \frac{(3-1)}{5}\right)$$

$$= 1.12 + 2.4\left(0.5426 - \frac{2}{5}\right)$$

$$= 1.46224$$

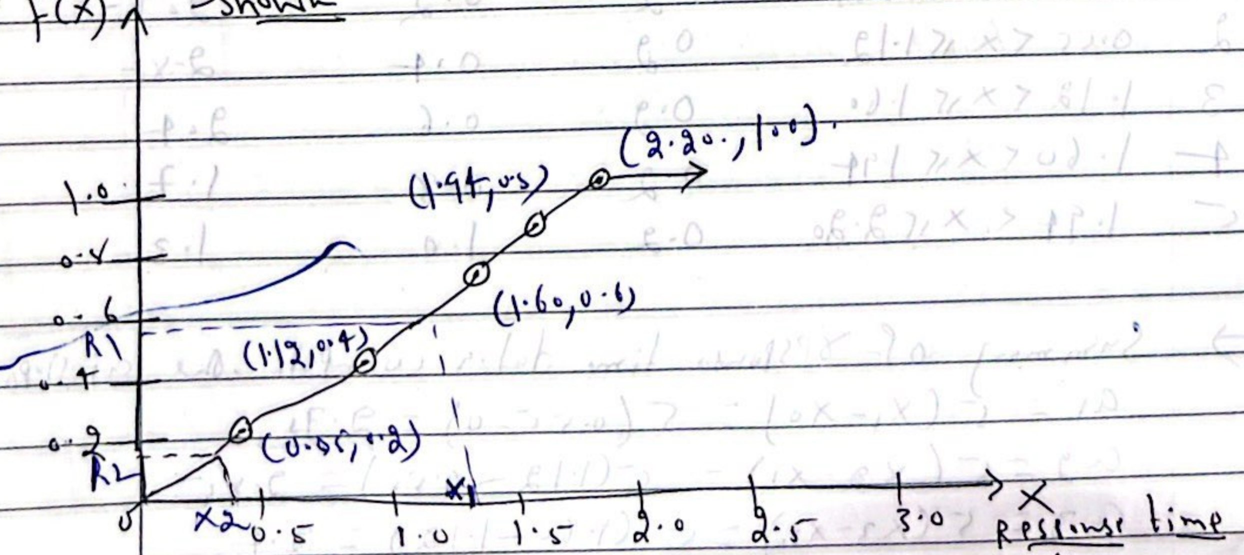
Similarly $R_2 = 0.1524$ which is lying between 0 and $1/5 = 0.2$ i.e first interval. Hence $i = 1$

$$\Rightarrow X_2 = x_{1-1} + a_1\left(R_2 - \frac{1-1}{5}\right)$$

$$= 0 + 2.75(0.1524 - 0)$$

$$= 0.4191$$

Graphically the process of generating the random variate is shown



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Discrete Distributions

All discrete distributions can be generated via the inverse xform technique. either numerically through a table look-up procedure or, in some cases algebraically. The final genr scheme being in terms of a formula.

A discrete uniform Distribution

consider discrete uniform distribution on $\{1, 2, \ldots k\}$ with pmf and CDF given by.

$$P(x) = 1/k \quad , \quad x = 1, 2, \ldots k.$$

$$F(x) = \begin{cases} 0 , & x < 1 \\ 1/k , & 1 \le x < 2 \\ 2/k , & 2 \le x < 3 \\ \vdots & \\ \frac{k-1}{k} , & k-1 \le x \le k \\ 1 , & k \le x. \end{cases}$$

→ Let $x_i = i$ and $r_i = p(1) + \cdots + p(x_i) = F(x_i) = i/k$ for $i = 1, 2, \ldots k$, if generated random number R satisfies.

$$r_{i-1} = \frac{i-1}{k} < R \le r_i = \frac{i}{k}$$

then X is generated by setting $X = i$. now above inequality can be solved for i,

$i-1 < RK \le i$.
$RK \le i < RK+1$.

Let $\lceil y \rceil$ denote smallest integer $\geq y$, for e.g $\lceil 7.82 \rceil = 8$, $\lceil 5.13 \rceil = 6$ and $\lceil -1.32 \rceil = -1$

for $y \geq 0$, $\lceil y \rceil$ is a function that rounds up. This notation and inequality yield a formula for generating X, namely

$X = \lceil RK \rceil$.

→ For e.g consider the generating of a random variate X that is uniformly distributed on $\{1, 2, --- 10\}$. The variate X might represent the no of pallets to be loaded onto a truck. Using Table A.1 as a source of random no R and using above Eqn with $k=10$ yields

$R_1 = 0.78$, $X_1 = \lceil 7.8 \rceil = 8$
$R_2 = 0.03$, $X_2 = \lceil 0.3 \rceil = 1$
$R_3 = 0.23$, $X_3 = \lceil 2.3 \rceil = 3$
$R_4 = 0.97$, $X_4 = \lceil 9.7 \rceil = 10$.

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* The Geometric Distribution *

Consider geometric distribution with pmf.

$P(x) = p(1-p)^x, \quad x = 0, 1, 2 \ldots$

Where $0 < p < 1$, its cdf is given by

$$F(x) = \sum_{j=0}^{x} p(1-p)^j$$

$$= \frac{p\{1-(1-p)^{x+1}\}}{1-(1-p)}$$

$$= 1-(1-p)^{x+1}$$

for $x = 0, 1, 2 \ldots$ using the inverse xform technique i.e inequality, that a geometric random variable X will assume the value x whenever:

$$F(x-1) = 1-(1-p)^x < R \leq 1-(1-p)^{x+1} = F(x)$$

Where R is a generated random number assumed $0 < R < 1$, solving the above eqn.

$$(1-p)^{x+1} \leq 1-R < (1-p)^x$$

$$(x+1)\ln(1-p) \leq \ln(1-R) < x\ln(1-p)$$

But $1-p < 1$ implies than $\ln(1-p) < 0$ so that.

$$\frac{\ln(1-R)}{\ln(1-p)} - 1 \leq x < \frac{\ln(1-R)}{\ln(1-p)}$$

Thus $X = x$ for that integer value of x satisfying inequality above in brief and using round up function $\lceil \cdot \rceil$, we have.

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$$X = \left\lceil \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rceil$$

Since p is fixed parameter, let $\beta = -1/\ln(1-p)$, then $\beta > 0$ and

$$X = \lceil -\beta \ln(1-R) - 1 \rceil$$

$\Rightarrow$ $-\beta \ln(1-R)$ is an exponentially distributed random variable with mean β, so one way of generating a geometric variate with parameter p is to generate an exponential variate with parameter $\beta^{-1} = -\ln(1-p)$, subtract one and round up.

$\Rightarrow$ Occasionally, there is needed a geometric variate X that can assume values $\{q, q+1, q+2, \ldots\}$ with pmf $p(x) = p(1-p)^{x-q}$ $(x = q, q+1, \ldots)$

Such a variable X can be generated from eqn.

$$X = q + \left\lceil \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rceil$$

One of most common cases is $q = 1$.

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Example: The CDF of a discrete random variable X is given by.

$$F(x) = \frac{x(x+1)(2x+1)}{n(n+1)(2n+1)}, \quad x = 1, 2 \ldots n.$$

Generate two values of X using random no. 0.38 and 0.75 and when $n=5$.

$\Rightarrow$ For a given random number R, X will take one value in $R_X = \{1, 2, 3, 4, 5\}$ provided that.

$$F(x_{i-1}) < R \leq F(x_i).$$

But $F(x) = \frac{x(x+1)(2x+1)}{n(n+1)(2n+1)}$.

Put $n=5$, we have.

$$F(x) = \frac{x(x+1)(2x+1)}{330}$$

$\rightarrow$ we know that.

$$F(x_{i-1}) < R \leq F(x_i).$$

$$= \frac{(x-1)(x)(2x-1)}{330} < R \leq \frac{x(x+1)(2x+1)}{330}.$$

As $x = 1, 2, \ldots 5$

$F(1) = (1)(2)(3)/330 = 0.018$

$F(2) = 2(3)(5)/330 = 0.091$.

$F(3) = 3 \cdot 4 \cdot 7/330 = 0.255$.

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$$F(4) = \frac{4(5)(9)}{330} = 0.546.$$

$$F(5) = \frac{5(6)(11)}{330} = 1.$$

Now X can be generated by table look-up method using table.

i	x_i	$F(x_i)$
1	1	0.018
2	2	0.091
3	3	0.255
4	4	0.546
5	5	1

The discrete random variate is.

$$X = i \text{ if } F(x_{i-1}) < R \leq F(x_i).$$

Now $R_1 = 0.38$ and $R_2 = 0.75$.

Now $[F(x_3) = 0.255] < [R_1 = 0.38] \leq [F(x_4) = 0.546]$.

$\Rightarrow$ $i = 4$ and $X_1 = x_4 = 4$

Similarly, $[F(x_4) = 0.546] < [R_2 = 0.75] \leq [F(x_5) = 1]$.

$\Rightarrow$ $i = 5$ and $X_2 = x_5 = 5$

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Example In a company group of particular items is accepted by performing certain quality control tests which will be approximated by geometric distribution on the range $\{X \geq 1\}$ with mean of 3 items, generate two values of X using random nos. 0.238 and 0.502

$\Rightarrow$ Given that geometric distribn is defined on the range $\{X \geq 1\}$.

$\therefore$ pmf is - $P(X) = P(1-P)^{X-1}$ for $X = 1, 2, \ldots$
with mean $1/P$.

$\rightarrow$ The geometric random variate is.

$$X_i = q + \left\lceil \frac{\ln(1-R_i)}{\ln(1-P)} - 1 \right\rceil$$

Given that mean $= 3 = 1/P$.
$\Rightarrow P = 1/3$ and $q = 1$.

and $R_1 = 0.238$ and $R_2 = 0.502$.

$$X_1 = 1 + \left\lceil \frac{\ln(1-R_1)}{\ln(1-\frac{1}{3})} - 1 \right\rceil$$

$$= 1 + \lceil -2.466 \ln(1-0.238) - 1 \rceil$$

$$= 1 + 0 = 1$$

$$X_2 = 1 + \lceil -2.466 \ln(1-0.502) - 1 \rceil$$
$$= 1 + 1 = 2$$

Marks Obtained

Convolution Method

→ The probability distribⁿ of a sum of two or more independent random variables is convolution of the distributions of variables.

→ This method is used when random variable X can be expressed as a sum of other random variables i.e ($X = X_1 + X_2 + \cdots X_n$) that are independent and identically distributed and easier to generate than X.

→ The assumption here is that $X_1 + X_2 + \cdots X_n$ has same distribution as X.

→ Hence we say that this method adds two or more random variables to get a new random variable with desired distribution.

→ This method is used to generate Erlang and binomial variates.

→ As $X = X_1 + X_2 + \cdots X_n$ and let X has distribⁿ function F and X_i has distribution function G.

1) Generate n number of X_i which are independent and identically distributed each with distribution function G.

2) Return $X = X_1 + X_2 + \cdots X_n$

→ This method is unambiguous provided that generatⁿ of each X_i is easy. However it is not the most efficient algorithm.

→ convolution method can be considered as a special case of the more general method of transforming

Some intermediate random variables into the desired variate.

⟹ * Erlang Distribution *

→ we know that an Erlang variable X with parameters k and θ is sum of k independent exponential random variables x_i $(i=1,2,\ldots k)$, each with mean $1/k\theta$

i.e. $X = \sum_{i=1}^{k} x_i$

→ As each x_i is exponentially distributed with mean $1/\lambda$, so it can be generated by inverse transform technique.

→ $x_i = -\frac{1}{\lambda} \ln R_i$

$$X = \sum_{i=1}^{k} -\frac{1}{\lambda} \ln R_i$$

→ $1/\lambda = 1/k\theta$

$$\therefore X = \sum_{i=1}^{k} -\frac{1}{k\theta} \ln R_i = -\frac{1}{k\theta} \ln \left\{ \prod_{i=1}^{k} R_i \right\}$$

→ From above eqn we say that k uniform random nos are required to generate a single Erlang variate.

Algorithm

1) Generate $x_1, x_2 \ldots x_k$ as independent and identically distributed exponential random variates.

2) compute desired random variate by

$$X = -\frac{1}{k\theta} \ln \left\{ \prod_{i=1}^{k} R_i \right\}$$

3) if k is large then Erlang variates are generated efficiently using acceptance rejection technique for gamma distribution.

⇒ Example The daily consumption of bread in a hostel is approximated by Erlang distribution with parameters $k = 3$ and $\theta = 10$. Generate two values of consumption from this distribution. Use $R_1 = 0.381$, $R_2 = 0.142$, $R_3 = 0.659$.

⇒ The Erlang random variate is

$$X = -\frac{1}{k\theta} \ln \left\{ \prod_{i=1}^{k} R_i \right\}$$

$k = 3$, $\theta = 10$ and

$$X = -\frac{1}{3(10)} \ln \left[(0.381)(0.142)(0.659) \right]$$

$$= \underline{0.11}$$

Acceptance- Rejection Technique

→ Suppose that analyst needed to devise a method for generating random variates, X, uniformly distributed between 1/4 and 1. One way to proceed would be to follow these steps –

Step 1 Generate random number R
Step 2 (a):- If $R \geq 1/4$ accept $X = R$ go to step 3.
Step 2(b):- If $R < 1/4$ reject R return to step 1.
Step 3 :- If another uniform random variate on $[1/4, 1]$ is needed repeat the procedure beginning at step 1, if not, stop.

→ Each time step 1 is executed, a new random number R must be generated. Step 2(a) is an "acceptance" and step 2(b) is "rejection".

→ To summarize the technique, random variates (R) with some distribution (here uniform on $[0,1]$) are generated until some condition $(R > 1/4)$ is satisfied. When the condition is finally satisfied the desired random variate, X (here uniform on $[1/4, 1]$) can be computed $(X = R)$. This procedure can be shown to be correct by recognizing that accepted values of R are conditioned values. R itself does not have the desired distribution but R conditioned on the event $\{R \geq 1/4\}$ does have the desired distribution. To show this, take $1/4 \leq a < b \leq 1$, then

$$P(a < R \leq b \mid 1/4 \leq R \leq 1) = \frac{P(a < R \leq b)}{P(1/4 \leq R \leq 1)} = \frac{b-a}{3/4}.$$

Which is correct probability for uniform distribution on $[1/4, 1]$, the above eqn says that probability distribn of R, given that R is between 1/4 and 1 (all other values of R thrown out) is the desired distribn ∴ if $1/4 \leq R \leq 1$ set $X = R$.

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Marks Obtained												

→ The efficiency of acceptance - rejection technique depends heavily on being able to minimize the no of rejections.

→ In this example, the Probability of rejection is $P(R < 1/4) = 1/4$ so that the no of rejections is a geometrically distributed random variable with Probability of "success" being $P = 3/4$ and mean no of rejections $(1/P - 1) = 4/3 - 1 = 1/3$

→ The mean no of random numbers R required to generate one variate X is one more than the number of rejections hence it is $4/3 = 1.33$

→ In other words to generate 1000 values of X would require approximately 1333 random nos R

→ In the present situation an alternate procedure exists for generating a uniform variate on $[1/4, 1]$ {uniform distrib} ~~which~~ in inverse xformation which reduces $X = a + (b-a)R = \frac{1}{4} + (1 - \frac{1}{4})R$
$= \frac{1}{4} + (3/4)R$.

∴ inverse xform is more effisient an alternate procedure.

→ However for some important distribn such as normal, gamma and beta, the inverse CDF does not exist in closed form and therefore inverse transform technique is difficult.

→ Acceptance- rejection technique is illustrated below for generation of random variates for Poisson, nonstationary Poisson and gamma distributions.

* **Poisson Distribution** *

→ A Poisson random variable N, with mean $\alpha > 0$ has pmf

$$P(n) = P(N=n) = \frac{e^{-\alpha}\alpha^{n}}{n!}, \quad n = 0, 1, 2 \ldots$$

⇒ N can be interpreted as no of arrivals from a Poisson arrival process in one unit of time.

⇒ For i.e. interarrival times $A_1, A_2 \ldots$ of successive customers are exponentially distributed with rate α (i.e α is the mean no of arrivals per unit time)

→ There is a relationship betwn the (discrete) Poisson and continuous exponential distribution.

$$N = n$$

if and only if.

$$A_1 + A_2 + \ldots + A_n \leq 1 < A_1 + \ldots + A_n + A_{n+1}$$

→ The above eqn says that $N = n$, says there were exactly n arrivals during one unit of time, but second relation says that nth arrival occured before time 1 while (n+1)th arrival occured after time 1 - clearly these two statements are equivalent.

→ Proceed now by generating exponential inter arrival times until some arrival say n+1 occur after time 1, then set $N = n$

→ For efficient generation purposes, above relation is usually simplified by using eqn exponential distribution eqn.

$$A_i = (-1/\alpha) \ln R_i \text{ to obtain.}$$

$$\sum_{i=1}^{n} -\frac{1}{\alpha} \ln R_i \geq 1 < \sum_{i=1}^{n+1} -\frac{1}{\alpha} \ln R_i$$

→ Multiply by $-\alpha$, which reverses the sign of inequality and use the fact that a sum of logarithms is the logarithm of a product to get.

$$\ln \prod_{i=1}^{n} R_i = \sum_{i=1}^{n} \ln R_i \geq -\alpha > \sum_{i=1}^{n+1} \ln R_i = \ln \prod_{i=1}^{n+1} R_i$$

Finally use relation $e^{\ln x} = x$ for any no x to obtain.

$$\prod_{i=1}^{n} R_i \geq e^{-\alpha} > \prod_{i=1}^{n+1} R_i$$

⇒ The procedure for generating a Poisson random variate N is given by following steps.

Step 1 set $n = 0$, $P = 1$

Step 2 Generate random no R_{n+1}, replace P by $P \cdot R_{n+1}$.

Step 3 :- if $P < e^{-\alpha}$, then accept $N = n$ otherwise reject the current n, increase n by one and return to step 2.

→ Notice that upon completion of step 2, P is equal to the right most expression in above eqn. The basic idea of rejection technique is again exhibited if $P \geq e^{-\alpha}$ in step (3) then n is rejected and generation process must proceed through at least one more trial.

→ How many random nos will be required, on the average to generate one Poisson variate N? if N = n, then n+1 random nos are required so the avg no is given by

$$E(N+1) = \alpha + 1$$

which is quite large if mean α of Poisson distribution is large.

Example Generate three Poisson variates with mean $\alpha = 0.2$ first compute $e^{-\alpha} = e^{-0.2} = 0.8187$. Next get sequence of random nos R from table A.1 and follow steps

Step 1 Set $n = 0$, $P = 1$.
Step 2:- $R_1 = 0.4357$, $P = 1 \cdot R_1 = 0.4357$.
Step 3:- Since $P = 0.4357 < e^{-\alpha} = 0.8187$ accept $N = 0$
Step 1-3. ($R_1 = 0.4146$ leads to $N = 0$)
Step 1 Set $n = 0$, $P = 1$.
Step 2 $R_1 = 0.8353$ $P = 1 \cdot R_1 = 0.8353$.
Step 3 Since $P \geq e^{-\alpha}$, reject $n = 0$ and return to Step 2 with $n = 1$.
Step 2 $R_2 = 0.9952$, $P = R_1 R_2 = 0.8313$.
Step 3 Since $P \geq e^{-\alpha}$ reject $n = 1$ and return to Step 2 with $n = 2$
Step 2 $R_3 = 0.8004$ $P = R_1 R_2 R_3 = 0.6654$
Step 3 Since $P < e^{-\alpha}$, accept $N = 2$

	7	8	9	10	11	12	Total
Marks Obtained							

The calculations required for generation of these three Poisson variates are summarized below.

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (accept)	$N=0$
0	0.4146	0.4146	$P < e^{-\alpha}$ (accept).	$N=0$
0	0.8353	0.8353	$P \geq e^{-\alpha}$ (reject).	
1	0.9952	0.8313	$P \geq e^{-\alpha}$ (reject).	
2	0.8004	0.6654	$P < e^{-\alpha}$ (accept),	$N=2$

It took five random nos R to generate three Poisson variate here ($N=0$, $N=0$ and $N=2$) but in long run, to generate say 1000 Poisson variate with mean $\alpha = 0.2$, it would require approximately $1000(\alpha+1)$ or 1200 random nos