Queuing: Models

1/17/20,000

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Example:- A tool crib has exponential interarrival and service times and serves a very large group of mechanics. The mean time between arrivals is 4 minutes. It takes 3 minutes on the average for a tool-crib attendant to service a mechanic. The attendant is paid \$10 per hour and the mechanic is paid \$15 per hour. Would it be advisable to have a second tool-crib attendant?

$\Rightarrow$ The tool crib is modeled by an M/M/c queue. ($\lambda = 1/4$, $\mu = 1/3$, $c = 1$ or 2).

$\rightarrow$ Mean cost per hr = \$10c + \$15L.

assuming that mechanics impose cost on the system while in the queue and in service.

case 1:- one attendant - M/M/1 ($c=1$, $\rho = \lambda/\mu = 0.75$)

$L = \rho/(1-\rho) = \underline{3 \text{ mechanics}}$

Mean cost per hr = \$10(1) + \$15(3) = \$55 per hr

case 2:- Two attendant - M/M/2 ($c=2$, $\rho = \lambda/c\mu = 0.3$

$L = c\rho + [(c\rho)^{c+1} P_0]/[c(c!)(1-\rho)^2]$

$= \underline{0.8727}$.

where

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} ((c\rho)^n / n!) \right] + \left[((c\rho)^c (1/c!)(1/(1-\rho)) \right] \right\}^{-1}$$

$$= \underline{0.4545}$$

→ Mean cost per hr = \$10 (2) + \$15 (0.8727)
= \$33.09 per hr.

It would be advisable to have a second attendant because long run costs are reduced by \$21.91 per hour.

Example :- At Metropolis city Hall, two workers "Pull strings" (make deals) every day. strings arrive to be pulled on an avg. of one every 10 minutes throughout the day. It takes an average of 15 minutes to pull a string. Both times between arrivals and service times are exponentially distributed. what is the probability that there are no strings to be pulled in the system at a random point in time? what is the expected number of strings waiting to be pulled? what is the probability that both string pullers are busy? what is the effect on performance if a third string-puller working at the same speed as the first two is added to the system?

⇒ M/M/2 queue ($\lambda = 1/10$, $\mu = 1/15$, $\rho = 0.75$)

⇒ a) The probability that there are no strings to be pulled is.

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} ((c\rho)^n / n!) \right] + \left[(c\rho)^c (1/c!) / (1-\rho) \right] \right\}^{-1}$$

$$= \underline{0.1429}$$

b) The expected number of strings waiting to be pulled is

$$L_q = [c(c\rho)^{c+1} P_0] / [c(c!)(1-\rho)^2]$$

$$= \underline{1.929} \text{ strings.}$$

c) The probability that both strings pullers are busy is

$$P(L(\infty) \geq 2) = [(c\rho)^2 P_0] / [(c!)(1-\rho)]$$

$$= \underline{0.643}$$

d) If a third string puller is added to the system (M/M/3 queue; $c=3$ $\rho = 0.50$), the measures of performance become.

$$P_0 = 0.2105, \quad L_q = 0.2368, \quad P(L(\infty) \geq 3)$$

$$= \underline{0.2368}.$$

Example:- Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially, with mean 3 minutes, find.

(imp)

i) the probability that an arrival finds that four persons are waiting for their turn.

ii) the average no of persons waiting and making telephone calls.

iii) the average length of the queue that is formed from time to time.

Solution

$\Rightarrow$ The average inter arrival time being 10 minutes, the average arrival rate $= 60/10 = 6$ customers per hr. Similarly average length of phone call being 3 minutes the avg service rate $= 60/3 = 20$ customers per hr.

$\Rightarrow$ with $\lambda = 6$ customers / hr and $\mu = 20$ cust/hr,

we have $\rho = \lambda/\mu = 6/20 = 0.3$

i) With four people waiting for their turn in the queue implies a total of 5 people in the system. Thus the probability that an arrival finds four persons in the queue is.

$$P(n=5) = \rho^5 (1-\rho)$$
$$= (0.3)^5 (1-0.3)$$
$$= 0.0017$$

ii) The average number of people waiting and making calls is given by expected length of the system thus,

$$L_s = \frac{\rho}{1-\rho} = \frac{0.3}{1-0.3} = 0.43$$

iii) The average length of queue that is formed from time to time

$$L_q' = 1/1-\rho = 1/1-0.3 = 1.43$$

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Example:- The service station has a central store where service mechanics arrive to take spare parts for the jobs they work upon. The mechanics wait in queue if necessary and are served on the first come first served basis. The store is manned by one attendant who can attend 8 mechanics in an hr on an average. The arrival rate of mechanics averages 6 per hr. Assuming that the pattern of mechanics arrivals is Poisson distributed and servicing time is exponentially distributed, determine Ws, Wq and Lq where the symbols carry their usual meaning.

⇒ Solution According to given information
$\lambda = 6$ mechanics / hr and $\mu = 8$ mechanics / hr.
∴ Avg utilisation $\rho = 6/8$, which is less than unity with these values

i) Expected time spent by a mechanic in the system
$W_s = 1/\mu-\lambda = 1/8-6 = 1/2$ hr $= 30$ min

ii) Expected time spent by a mechanic in the queue
$W_q = \lambda/\mu(\mu-\lambda) = 6/8(8-6) = 6/16$ hr
$= 22.5$ min

iii) Expected no of mechanics in the queue.
$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{6^2}{8(8-6)} = \frac{36}{16} = 2.25 \text{ mechanics}$$

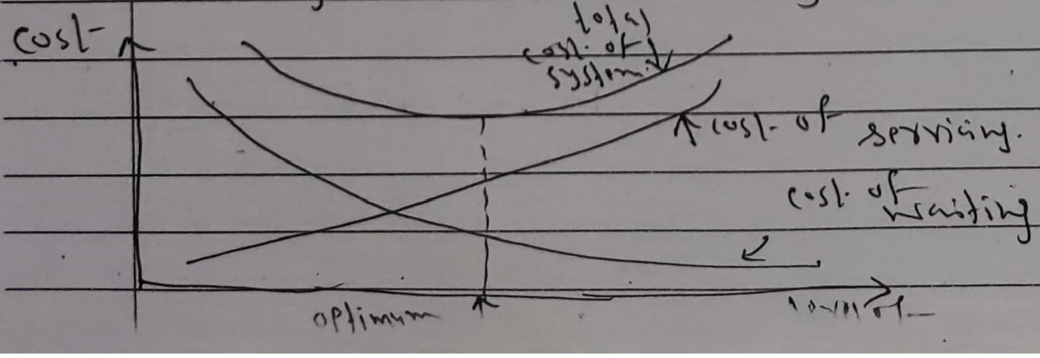
Cost Analysis

The information provided by queuing model can be usefully employed in determining the appropriate level of service.

→ For instance, the manager of the station might be concerned about the time that the mechanics have to wait in the queue in order to obtain the spare parts or tools that they need, because it costs to the service station (to pay for their idle time). The manager may consider employing additional attendants in the storeroom so that service may be provided at the faster rate and consequently the mechanics don't have to wait long for obtaining supplies.

→ This decision would have the twin effects, first it reduces the idle time and therefore the cost (associated with it) and second it increases the cost of providing service through payments to the additional hand employed.

→ The manager would do well to employ additional attendants if and so long as the saving in cost more than offsets the increase in cost. The optimum level of service would naturally be determined where the total of the waiting time cost and cost of providing service is the minimum. ~~These~~ Shown in fig below, where it clear that increasing the service level would result in increasing the cost of service and reducing the cost of waiting time.



⇒ Let us ~~cautious~~ analytically consider the manager's problem. Suppose that service mechanics are paid at Rs 8 per hr and storeroom attendant's wages are Rs 5 per hr. In a typical 8-hr day the total arrivals would be $8 \times 6 = 48$. We know that an arriver has to wait 1/2 hr before he obtains parts of his requirement. Thus, the total waiting time in the system for a day equals $48 \times 1/2 = 24$ hrs

⇒ Now, the daily cost of service provided by one attendent = cost of service + cost of waiting.

→ cost of service = No of attendants × hourly rate × No of hrs.

$= 1 \times 5 \times 8 =$ Rs 40.

→ cost of waiting = No of hrs waiting × hourly rate. (for mechanics)

$= 24 \times 8 = 192$

∴ Total cost $= 40 + 192 =$ Rs. 232 per day

→ Suppose now that the manager believes that i one more attendant is hired for the store room. The service rate can be increased from 8 to 12 mechanics an hr. The expected time spent by a mechanic in the system would equal.

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-6} = 1/6 \text{ hr}.$$

→ Total waiting time in the system for a day

$48 \times 1/6 = 8$ hrs

Accordingly.

Service cost $= 2 \times 5 \times 8 =$ Rs 80

Cost of waiting $= 8 \times 8 =$ Rs 64

∴ Total cost $= 80 + 64 =$ Rs 144 per day.

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Thus the total cost of providing services reduces from Rs 232 to Rs 144 per day by hiring an additional attendent.

→ Further suppose that manager believes that by employing yet another person the service rate can be stepped up to 16 mechanics per hr. with this.

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{16-6} = 1/10 \text{ hr}$$

→ Total waiting time in the system $= 48 \times \frac{1}{10} = 4.8$ hrs per day

Service cost $= 3 \times 5 \times 8 =$ Rs 120.
waiting cost $= 4.8 \times 8 =$ Rs 38.4

∴ Total cost = Rs 158.4.

clearly it is not worthwhile employing the third person. To minimise the total cost.
∴ the manager should employ two attendants.

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Example:- A repairman is to be hired by a company to repair machines that break down following a Poisson process with an average rate of four per hr. The cost of non productive machine time is Rs 90 per hr. The company has the option of choosing either a fast or a slow repairman. The fast repairman charges Rs 70 per hr and will repair machines at an avg rate of 7 per hr while the slow repairman charges Rs 50 per hr and will repair machines at an avg rate of 6 per hr. which repairman should be hired?

⟹ For solving this problem we compare the total expected daily cost for both the repairmen. Then would equal the total wages paid plus the cost of non-productive machine hrs.

→ We have. Total wages = hourly rate × No. of hrs.

For fast repairman

Total wages = 70×8 (assuming 8-hr shift).
= Rs 560.

For slow - Total wages = 50×8 = Rs 400.

cost of non productive time can be calculated as under

→ Expected No of machines in the system × cost of idle machine hr × No of hrs

→ Expected no of machines $L_s = \lambda / \mu - \lambda$.

→ cost of idle machine = Rs 90/hr.

→ No of hrs (assuming 8-hr shift) = 8.

→ For fast repairman.

$\lambda = 4$ machines/hr and $\mu = 7$ machines/hr.

$L_s = \frac{4}{7-4} = 4/3$ machines.

thus cost of non productive machine time =

$4/3 * 90 * 8 =$ Rs 960.

→ For slow repairman

$\lambda = 4$ machines/hr and $\mu = 6$ machines/hr.

$L_s = 4/6-4 =$ 2 machines.

Accordingly cost of non productive machine time =

$2 \times 90 \times 8 =$ Rs 1440.

⇒ The cost of non productive machine time can be alternatively calculated as follows.

⇒ Expected time a machine spends in the system × No of arrivals per day × cost of idle time per hr.

⇒ The product of first two elements gives the total m/c hrs lost per day which when multiplied by the hourly rate yields the cost of idle (non productive) time per day.

W_s (for fast) $= \frac{1}{7-4} = 1/3$ hr.

W_s (for slow) $= 1/6-4 = 1/2$ hr.

→ No of arrivals per day of 8 hrs $= 8 \times 4$
$= 32$

→ cost of idle time per hr = Rs 90.

→ cost of idle time is.
for fast repairman $= 1/3 \times 32 \times 90$.
$=$ Rs 960

→ for slow repairman $= 1/2 \times 32 \times 90$
$=$ Rs 1440.

∴ Total cost
Fast — Rs 560 + Rs 960 = Rs 1520.
Slow — Rs 400 + Rs 1440 = Rs 1840.

∴ Fast repairman should be employed by the manager.

Example:- Customers arrive at First class ticket counter of a theatre at a rate of 12 per hr. There is one clerk serving the customers at a rate of 30 per hour.

i) what is the probability that there is no customer in counter (i.e that the system is idle)?

ii) what is probability that there are more than 2 customers in the counter?

iii) what is the probability that there is no customer waiting to be served?

iv) what is probability that a customer is being served and nobody is waiting?

⇒ $\lambda = 12$ customers / hr
$\mu = 30$ customers / hr.
$\rho = \lambda/\mu = 12/30 = 0.4$.

i) $P(\text{System is idle}) = P(0) = 1 - \rho = 1 - 0.4 = 0.6$.

ii) $P(n > 2) = 1 - P(n \leq 2)$.

$= 1 - [P(0) + P(1) + P(2)]$

$= 1 - [0.6 + 0.4 \times 0.6 + 0.4^2 \times 0.6]$.

$= 1 - (0.6 + 0.24 + 0.096)$

$= 1 - 0.936 = \underline{\underline{0.064}}$

Also. $P(>n) = \rho^{n+1}$

$\therefore P(>2) = 0.4^{2+1} = 0.4^3 = \underline{0.064}$

iii) P(no customers waiting to be served)

$= P(0) + P(1) = 0.6 + 0.24 = \underline{0.84}$

iv) P(a customer is being served and none is waiting)

$= \underline{P(1)} = \underline{0.24}$

Example 1. The rate of arrival of customers at a public telephone follows Poisson distribution, with an avg time of ten minutes between one customer and the next. The duration of a phone call is assumed to follow exponential distribution with a mean time of three minutes.

i) What is the probability that a person arriving at the booth will have to wait?

ii) What is avg length of the queue?

iii) The MTNL will install another booth when it is convinced that the customers would have to wait for at least three minutes for their turn to make a call. How much should be the flow of customers in order to justify a second booth?

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⟹ arrival rate, $\lambda = 6$ customers / hr.
service rate, $\mu = 20$ customers / hr.

$\therefore \rho = \lambda/\mu = 6/20 = 0.3$.

i) A customer arriving at the booth will have to wait if booth is busy thus.

P(a customer has to wait) $= \rho = 0.3$.

ii) Average length of queue.

$L_q = \rho^2 / 1-\rho$

$= (0.3)^2 / 1-0.3 = \frac{0.09}{0.70} = 9/70$ customers

iii) $W_q = \lambda / \mu(\mu-\lambda)$.

$= \frac{6}{20(20-6)} = 3/140$ hr or 9/7 minute.

Let the new arrival rate be λ'. Setting $W_q =$ 3 minutes or 3/60 hr, we get:

$\frac{3}{60} = \frac{\lambda'}{20(20-\lambda')}$

or $20-\lambda' = \lambda'$ or $\lambda' = 10$ customers / hr

Thus, as soon as the arrival rate increases to 10 customers/hr, another booth may be installed.

Example:- A warehouse has only one loading dock manned by a three person crew. Trucks arrive at the loading dock at an avg rate of 4 trucks per hr. and arrivals are Poisson distributed. The loading of a truck takes 10 minutes on an avg and the loading time can be assumed to be exponentially distributed about this avg. The operating cost of a truck is Rs 100 per hr and the members of the loading crew are paid at a rate of Rs 25 per hr. Assuming that the addition of new crew members would reduce the loading time to 7.5 minute Would you advise the truck owner to add another crew of three persons ?

$\Rightarrow$ $\lambda = 4$ trucks/hr.
$\mu = 6$ trucks/hr.

We have:
Total hourly cost = Loading crew cost + cost of waiting time.

$\rightarrow$ Loading crew cost = No of loaders $\times$ hourly wage rate.
= 3×25
= Rs 75 per hr

$\rightarrow$ cost of waiting time = Expected waiting time per truck (W_s) $\times$ Expected arrivals per hr (λ) $\times$ hourly waiting cost
$= \frac{1}{6-4} \times 4 \times 100$ = Rs 200 per hr

Alternatively, cost of waiting time = Expected no of trucks in system (L_s) $\times$ hourly waiting cost.
$= \frac{4}{6-4} \times 100$
= Rs 200 per hr.

Total cost = Rs 75 + Rs 200
= Rs 275 per hr

→ With proposed crew addition.
Loading crew cost = 6 × 25 = Rs 150 per hr
Cost of waiting time = $\frac{4}{8-4} \times 100$
= Rs 100 per hr.

60/7.5 = 8 loads/hr

Total cost = Rs 150 + Rs 100 = Rs 250 per hr

Conclusion :- It is advisable to add a crew of three loaders.

Example: A typist at an office receives on an average 22 letters per day of typing. The typist works 8 hrs a day and it takes on an avg 20 minutes to type a letter. The company has determined that the cost of letter ~~writing~~ waiting to be mailed (opportunity cost) is 80 Paise per hr and the equipment operating cost plus the salary of the typist will be Rs 40 per day.

i) What is the typist utilisn rate?
ii) What is the average no of letters waiting to be typed?
iii) What is the avg waiting time needed to have a letter typed?
iv) What is the total cost of waiting letters to be mailed.

b) Forced to improve the letter typing service, the above company is planning to take lease of one of the two models of an automated typewriter available in the market. The daily cost and the resulting increase in typist efficiency are displayed in the table below.

Model	Additional cost per day	Increase in Typist Efficiency
I	R.s 37	50 %
II	R.s 39.	75 %.

What action should the company take to minimise the total daily costs of waiting letters to be mailed?

⟹ Solution

Arrival rate $\lambda = 22$ letters / day.

Service rate $\mu = \frac{8 \times 60}{20} = 24$ letters / day

i) Typist's utilisation rate $\rho = \lambda / \mu = 22/24$

$= 0.917$

ii) Expected no of letters waiting to be typed

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{22^2}{24(24-22)}$$

$= 10.08$ letters

iii) Expected time needed to have a letter typed.

$W_s = 1/\mu-\lambda.$

$= 1/24-22 = 1/2$ day or 4 hrs.

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iv) Total daily cost of waiting letters to be mailed = Expected no of letters in the system (L_s) x opportunity cost + salary of typist

Now $L_s = \lambda / \mu - \lambda$.

$$= \frac{22}{24-22} = 11 \text{ letters}$$

→ opportunity cost to mail a letter is 80 Paise per hr or 8 x 80 Paise = Rs 6.40 Per day

∴ Total opportunity cost = 6.40 x 11 = Rs 70.40

∴ Total daily cost = Rs 70.40 + Rs 40.
= Rs 110.40

b) Model I → Increase in efficiency is given to be 50% with this model. Therefore, Service rate = 24 x 1.5 = 36 letters per day

with λ = 22 letters Per day.

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{22}{36-22} = \frac{11}{7} \text{ letters.}$$

Now. Total daily cost = L_s x Opportunity cost + equipment operating cost + salary of typist

$$= \frac{11}{7} \times 6.40 + 37 + 40$$

$$= Rs\ 87.06$$

Model II With a 75% increase in service rate with this model, we have $\mu = 24 \times \frac{7}{4} = 42$ letters.

per day The given arrival rate being 22 letters per day, we get

$$L_s = \frac{22}{42-22} = \frac{11}{10} \text{ letters}$$

$$\therefore \text{Total daily cost} = \frac{11}{10} \times 6.40 + 39 + 40$$

$$= Rs\ 86.04$$

∴ It is advisable to use model II typewriter in order to minimise the total daily cost of waiting letters to be mailed.

Example:- A factory operates for 8 hrs. everyday and has 240 working days in the year. It buys a large number of small machines which can be serviced by its maintenance engineer at a cost of Rs 4 per hr for the labour and spare parts. The machines can alternatively be serviced by supplier at an annual contract price of Rs 20,000 including the labour and spare parts needed. The supplier undertakes to send a repairman as soon as a call is made but in no case more than one repair man is sent. The service times of the maintenance engineer and the supplier?

repairman are both exponentially distributed with respective means of 1.7 and 1.5 days. The machine breakdowns occur randomly and follow Poisson distribution with an avg of 2 in 5 days. Each hour that a machine is out of order, it costs the company Rs 8, which servicing alternative would you advise it to opt for?

⟹ Solution :- We shall calculate the total cost involved for each of the alternatives for taking the decision.

Total annual cost = cost of idle machinery time + cost of labour and spare parts

→ cost of idle Machinery Time = Expected no of machines in system × No of working hrs per year × Hourly idle time cost

Alternative I :- Maintenance by company engineer.

$\lambda = 2/5$ machines per day.

$\mu = 1/1.7$ machines per day.

→ Expected no of machines in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{2/5}{\frac{1}{1.7} - \frac{2}{5}} = \frac{17}{8} \text{ machines.}$$

→ cost of idle Machinery Time $= \frac{17}{8} \times 1920 \times 8$

$= \text{Rs } 32,640$

→ cost of labour and spare parts = No of machines in the system × No of working hrs per year × Hourly cost

$= \frac{17}{8} \times 1920 \times 4 = \text{Rs } 16,320$

$\therefore$ Total cost $= 32,640 + 16,320$.

$= $ R.s 48,960 p.a

Alternative 2:- Maintenance by Supplier

we have

$\lambda = 2/5$ machines per day.

$\mu = 1/1.5$ machines per day.

→ Expected no of machines in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{2/5}{\frac{1}{1.5} - \frac{2}{5}} = \frac{3}{2} \text{ machines}$$

→ cost of idle Machine Time $= \frac{3}{2} \times 1920 \times 8$

$= $ R.s 23,040.

→ cost of labour and spare parts $=$ R.s 20,000

$\therefore$ Total cost $= 23,040 + 20,000$

$= $ R.s 43,040 p.a

→ clearly then the maintenance of machines by the supplier is the better alternative.

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Example 1- On an average, 4 Patients per hour require the service of an emergency clinic. Also, on an average a patient requires 10 minutes of active attention. The clinic can handle only one emergency at a time. Suppose that it costs the clinic R.s 300 per patient treated to obtain an avg servicing time of 10 minutes and that each minute of decrease in this average time would cost R.s 50 per Patient treated. How much would have to be budgeted by the clinic to decrease the average size of queue from 4/3 Patients to 1/2 patients ?

⇒ Solution 1-

Mean arrival rate = $\lambda = 4$ Patients / hr.

Mean service rate $\mu = 6$ patients / hr.

$\therefore \rho = 4/6 = 2/3$.

→ At present expected length of queue.

$$L_q = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-2/3} = \frac{4/9}{1/3}$$

$$= 4/3 \text{ Patients}$$

→ To obtain the planned avg queue length, Let the service rate be μ', with this.

$$L_q = \frac{\lambda^2}{\mu'(\mu' - \lambda)}$$

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Subsidituting the known values,

$$\frac{1}{2} = \frac{4^2}{\mu'(\mu'-4)}$$

$$\mu'^2 - 4\mu' = 32$$

$$\text{or } \mu'^2 - 4\mu' - 32 = 0.$$

$$\therefore (\mu'-8)(\mu'+4) = 0.$$

We consider only +ve value. $\therefore \mu' = 8$ customers/hr.

→ The service rate of 8 customers per hour implies a service time equal to 7.5 minute per customer.

→ Thus, reduction in service time =

$$10 - 7.5 = \underline{2.5 \text{ min}}$$

∴ cost of reduction being Rs 50 per minute, the budget requirement equals

$$300 + 50 \times 2.5 = \underline{425 \text{ rupees per patient}}$$

∴ Total budget $= 4 \times 24 \times 425$

$= \underline{\text{Rs } 40{,}800 \text{ per day}}$

Example 1 – The tooth care hospital provides free dental service to the patients on every Sat-morning. There are three dentists on duty, who are equally qualified and experienced. It takes on an average 20 minutes for a patient to get treatment and actual time taken is known to vary approximately exponentially around this average. The patients arrive to the Poisson distribution with an average of 6 per hr. The administrative officer of the hospital wants to investigate the following.

a) The expected no of patients waiting in queue.
b) The average time that a patient spends at clinic
c) The avg percentage idle time for each of dentists.
d) The fraction of time at least one dentist is idle.

Solution

⟹ For the given information
$k = 3$, $\lambda = 6$ patients per hour,

$\mu = 1/20$ patients per min or $\underline{3 \text{ patients per hr.}}$

Thus. $\rho = 6/(3 \times 3) = \underline{2/3}$

Further,

$$P_0 = \left[1 + 2 + \frac{2^2}{2!} + \frac{3(2)^3}{3!(3-2)}\right]^{-1} = \frac{1}{9}.$$

Now,
a) Expected no of patients waiting in queue.

$$L_q = \frac{(\lambda/\mu)^k \rho}{k!(1-\rho)^2} \times P_0.$$

$$= \left[\frac{2^3 \times (2/3)}{3!\left(1-\frac{2}{3}\right)^2}\right]\left[\frac{1}{9}\right] = \frac{8}{9} = \underline{0.889.}$$

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b) Expected time a patient spends in system

$$W_s = W_q + \frac{1}{\mu}$$

here $W_q = L_q / \lambda = \frac{8}{9} \times \frac{1}{6} = \frac{4}{27}$.

$$W_s = \frac{4}{27} + \frac{1}{3} = \frac{13}{27} \text{ hrs.}$$

or $\underline{28.9 \text{ min}}$

c) To determine the percentage of idle time of dentists we realise that when the system is empty all the three dentists would be idle; when only one person is in the clinic, then two of them are idle; when two patients are in the clinic, only one doctor is idle; and finally when three or more patients are in the clinic, then all the dentists are busy. Assuming random selection of dentists, when more than one dentist is idle,

$$P(\text{idle dentist}) = 1 \times P_0 + \frac{2}{3} P_1 + \frac{1}{3} P_2$$

$$= \frac{1}{9} + \left(\frac{2}{3}\right)\left(\frac{2}{1!}\right)\left(\frac{1}{9}\right) + \left(\frac{1}{3}\right)\left(\frac{2^2}{2!}\right)\left(\frac{1}{9}\right)$$

$$= \frac{1}{3}.$$

Thus one third of his time a dentist has no patients to examine. Of course this probability can be directly calculated as equal to $1 - 1 = 1 - 2/3 = \underline{1/3}$.

d) Probability of at least one idle dentist

$$= \sum_{n=0}^{s-1} P_n$$

$$= P_0 + P_1 + P_2 = \underline{5/9}$$

8/17/20,000

K. J. SOMAIYA COLLEGE OF ENGINEERING

(Autonomous College Affiliated to University of Mumbai)

Candidate Roll No. ________ (In figures)

Test Exam.

Name : ________

Date : ________ 20

Examination : ________ Branch/Semester ________

Subject : ________

Junior Supervisor's full Signature with Date

Question No.	1	2	3	4	5	6	7	8	9	10	11	12	Total
Marks Obtained													

Q20:- Let x be defined as lifetime of battery. Then X is exponential ($\lambda = 1/48$) with CDF.

$$F(x) = 1 - e^{-x/48}, \quad x > 0.$$

a) The probability that battery will fail within one next twelve months, given that it has operated for sixty months is.

$$P(X \le 72 \mid X > 60) = P(X \le 12)$$
$$= F(12) = \underline{0.2212}$$

due to memoryless property.

b) Let Y be defined as the year in which the battery fails, then

$$P(Y = \text{odd year}) = (1 - e^{-0.25}) + (e^{-0.50} - e^{-0.75}) + \cdots$$
$$P(Y = \text{even year}) = (1 - e^{-0.50}) + (e^{-0.75} - e^{-1}) + \cdots$$

so, $P(Y = \text{even year}) = e^{-0.25} P(Y = \text{odd year})$,

$P(Y = \text{even year}) + P(Y = \text{odd year}) = 1$ and.

$e^{-0.25} P(Y = \text{odd year}) = 1 - P(Y = \text{odd year})$.

The probability that the battery fails during an odd year is

$$P(Y = \text{odd year}) = 1/(1 + e^{-0.25}) = \underline{0.5622}.$$

c) Due to memoryless property of exponential distribution the remaining expected lifetime is 48 months.

Q21:-

⇒ Service time, x_i is exponential ($\lambda = 1/50$) with CDF
$F(x) = 1 - e^{-x/50}$, $x > 0$.

a) The probability that two customers are each served within one minute is.

$$P(X_1 \leq 60, X_2 \leq 60) = [F(60)]^2$$
$$= (0.6988)^2$$
$$= \underline{0.4883}$$

b) The total service, $X_1 + X_2$, of two customers has an Erlang distribution (assuming independence). with CDF

$$F(x) = 1 - \sum_{i=0}^{1} [e^{-x/50} (x/50)^i / i!] \quad x > 0.$$

→ The probability that the two customers are served within two minutes is.

$$P(X_1 + X_2 \leq 120) = F(120) = \underline{0.6916}.$$

Q27:-

⟹ Let X represents the time until a car arrives, using Erlang distribution with.

$K\theta = 4, \; X = 1$

∴ desired Probability is given by.

$$F(1) = 1 - \sum_{i=0}^{2} \frac{e^{-4(1)} [4(1)]^i}{i!} = \underline{0.762}$$

Q 29:-

⟹ Let x be defined as grading time of all six Problems, then x is Erlang ($K = 6$, $\theta = 1/180$) with.

CDF.

$$F(x) = 1 - \sum_{i=0}^{6-1} [e^{-x/30} (x/30)^i / i!], \quad x > 0.$$

a) The Probability that grading is finished in 150 minutes or less is.

$$P(X \leq 150) = F(150) = \underline{0.3840}$$

b) The most likely grading time is the mode.

$= (K-1)/K\theta = \underline{150 \text{ minutes}}$

c) The expected grading time is.

$E(X) = \cancel{\theta} \; 1/\theta = \underline{180 \text{ minutes}}.$

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Q30:-

⇒ Let x be defined as life of a dual hydraulic system consisting of two sequentially activated hydraulic systems each with a life. Y, which is exponentially distributed ($\lambda = 2000$ hrs). Then X is Erlang ($k = 2$, $\theta = 1/4000$) with CDF.

$$F(x)' = 1 \sum_{i=0}^{2-1} \left[e^{-x/2000} (x/2000)^i / i! \right], \quad x > 0.$$

a) The Probability that the system will fail within 2500 hrs is.

$$P(X \leq 2500) = F(2500) = 0.3554$$

b) The Probability of failure within 3000 hrs is.

$$P(X \leq 3000) = F(3000) = 0.4424$$

if inspection is moved from 2500 to 3000 hrs, the Probability that the system will fail increases by 0.087.