Engineering Mechanics

Module 2.1 – **Kinematics of Particles and Rigid Bodies**

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Brief Contents of module 2.1

- Variable motion, motion curves (a-t, v-t, s-t)
  (acceleration curves restricted to linear acceleration only)
- Motion along plane curved path,
- Velocity & acceleration in terms of rectangular components,
- Tangential & normal component of acceleration,
- Relative velocities

Introduction

Dynamics includes:

- ***Kinematics***: study of the geometry of motion. Relates displacement, velocity, acceleration, and time ***without reference*** to the cause of motion. (i.e. *regardless of forces*).
- *Kinema* means movement in Greek
- Mathematical description of motion
  - Position
  - Time Interval
  - Displacement
  - Velocity; absolute value: speed
  - Acceleration
  - Averages of the latter two quantities.
- ***Kinetics***: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion

Introduction (Cont..)

Particle kinetics includes:

- ***Rectilinear motion***: position, velocity, and acceleration of a particle as it moves along a straight line.
- ***Curvilinear motion***: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.
- Please Recall
  1. Newton's three laws of motion
  2. Position, Displacement, velocity, acceleration
  3. Horizontal motion
  4. Vertical motion

Introduction (Cont..)

- Rectilinear Motion: Position, Velocity & Acceleration

  - ***Rectilinear motion:*** particle moving along a straight line

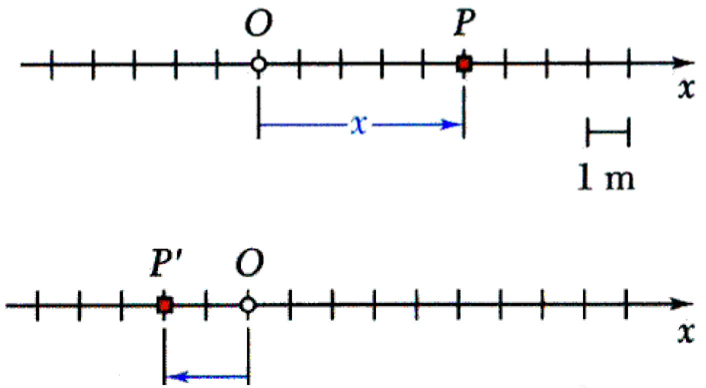
  - ***Position coordinate:*** defined by positive or negative distance from a fixed origin on the line.

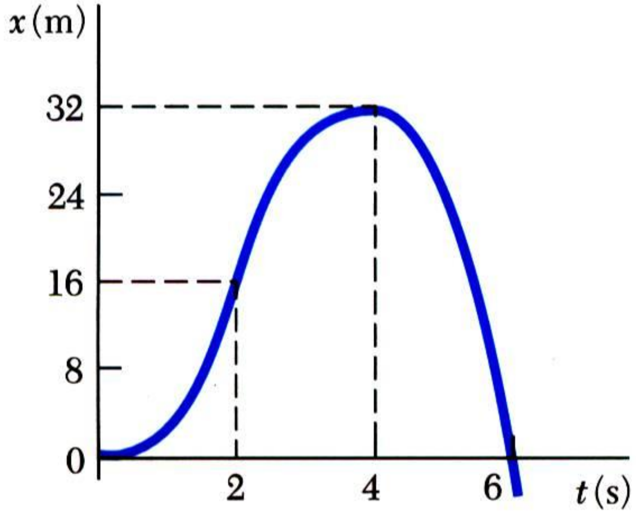
- The ***motion*** of a particle is known if the position coordinate for particle is known for every value of time *t*.

- or in the form of a graph *x* vs. *t*.

- May be expressed in the form of a function, e.g.,

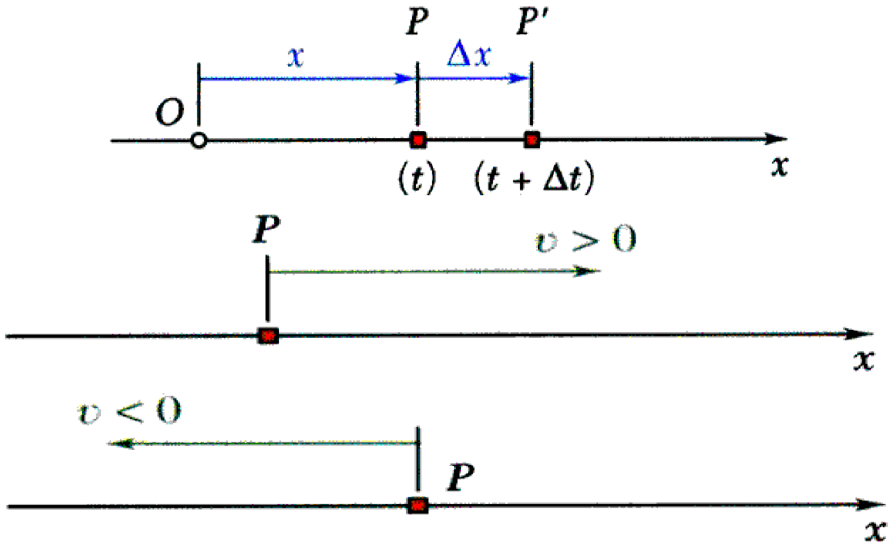
$$x = 6t^2 - t^3$$





Introduction (Cont..)

- Rectilinear Motion: Position, Velocity & Acceleration



- Consider particle which occupies position *P* at time *t* and *P'* at *t* + *Δt*,

$$\textit{Average velocity} = \frac{\Delta x}{\Delta t}$$

$$\textit{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

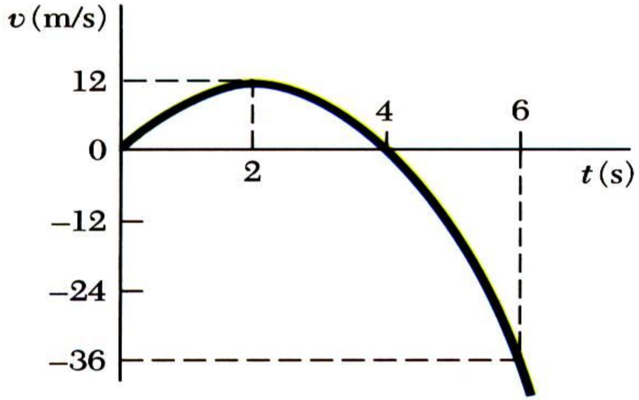
  - Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as *particle speed*.

  - From the definition of a derivative,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

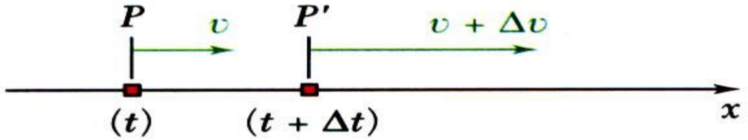
e.g., $x = 6t^2 - t^3$

$$v = \frac{dx}{dt} = 12t - 3t^2$$



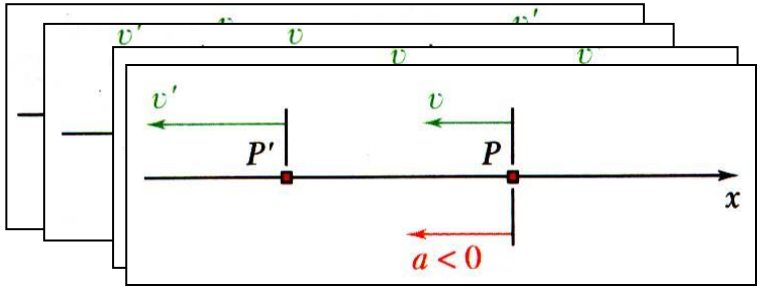
Introduction (Cont..)

- Rectilinear Motion: Position, Velocity & Acceleration



- Consider particle with velocity *v* at time *t* and *v'* at *t+Δt*,

$$\textit{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

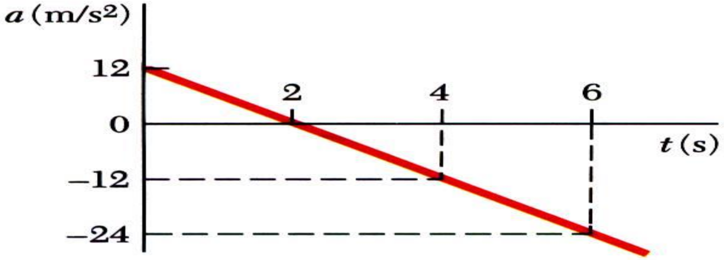


- Instantaneous acceleration may be:
  - positive: increasing positive velocity or decreasing negative velocity
  - negative: decreasing positive velocity or increasing negative velocity.
- From the definition of a derivative,

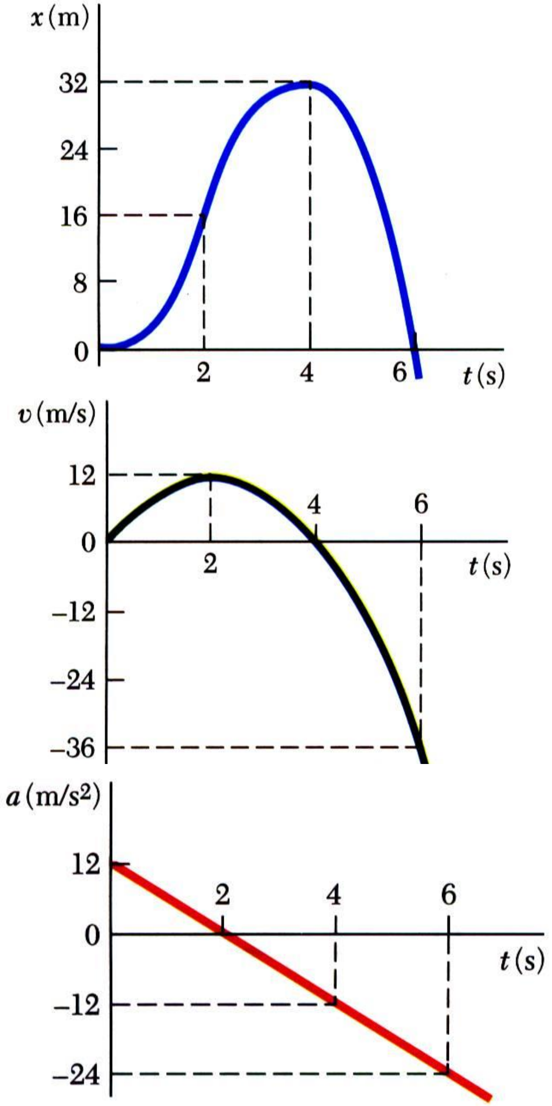
$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

e.g. $v = 12t - 3t^2$

$$a = \frac{dv}{dt} = 12 - 6t$$



Rectilinear Motion: Position, Velocity & Acceleration



- From our example,

$$x = 6t^2 - t^3$$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t$$

- What are x, v, and a at t = 2 s ?

Ans: at t = 2 s, x = 16 m, $v = v_{max}$ = 12 m/s, a = 0

- Note that v_{max} occurs when a=0, and that the slope of the velocity curve is zero at this point.

- What are x, v, and a at t = 4 s ?

Ans: at t = 4 s, $x = x_{max}$ = 32 m, v = 0, a = -12 m/s^2

One minute break

- **What is true about the kinematics of a particle?**

a) The velocity of a particle is always positive
b) The velocity of a particle is equal to the slope of the position-time graph
c) If the position of a particle is zero, then the velocity must zero
d) If the velocity of a particle is zero, then its acceleration must be zero

Determination of the Motion of a Particle

- **Generally we have three classes of motion**
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = f(x)$
  - acceleration given as a function of *velocity*, $a = f(v)$

- **If the acceleration is given, we can determine velocity and position by two successive integrations.**

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

$$a = \frac{d^2x}{dt^2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Rectilinear motion

For a particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

$$\frac{dx}{dt} = v = \text{constant}$$

$$\int_{x_0}^{x} dx = v \int_{0}^{t} dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt$$

If forces applied to a body are constant (and in a constant direction), then you have uniformly accelerated rectilinear motion.

Uniformly Accelerated Rectilinear Motion

For a particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant. You may recognize these constant acceleration equations from Physics courses.

$$\frac{dv}{dt} = a = \text{constant} \qquad \int_{v_0}^{v} dv = a\int_{0}^{t} dt \qquad v = v_0 + at$$

$$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_{0}^{t} \left(v_0 + at\right) dt \qquad x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

$$v\frac{dv}{dx} = a = \text{constant} \qquad \int_{v_0}^{v} v\, dv = a\int_{x_0}^{x} dx \qquad v^2 = v_0^2 + 2a\left(x - x_0\right)$$

Rectilinear Motion

- Velocity as a Function of Time

Integrate

ac = dv/dt, assuming that initially v= v0 when t = 0.

$$\int_0^v dv = \int_0^t a_c dt$$

$$v = v_0 + a_c t$$

Constant acceleration

- Position as a Function of Time

Integrate

v = ds/dt = v0 + act, assuming that initially *s* = *s0 when t = 0*

$$\int_0^s ds = \int_0^t (v_0 + a_c t) dt$$

$$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$$

Constant acceleration

- Velocity as a Function of Position

Integrate

v dv = a$_c$ *ds, assuming that initially v = v*$_0$ *at s = s*$_0$

$$\int_{v_0}^{v} v dv = \int_{s_0}^{s} a_c ds$$

$$v^2 = v_0^2 + 2a_c(s - s_0)$$

Constant acceleration

Motion with variable acceleration:

- The governing equations are

$$V = dx/dt, \quad a = dv/dt, \quad a = V.dv/dt$$

- ~~Motion under gravity~~

1. ~~Motion in vertical direction is influenced by gravitational force~~
2. ~~Acceleration of particle remains constant and equal to g (gravitational force)~~
3. ~~Acceleration due to gravity is directed towards centre of earth~~
4. ~~It is taken as negative (ve)~~
5. ~~It is a special case of uniformly accelerated motion hence equation of UAM are used with a = -g and s = y~~

Summary

Procedure:

1. Establish a coordinate system & specify an origin
2. Remember: x, v,a,t are related by:

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

$$a = \frac{d^2x}{dt^2}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

3. When integrating, either use limits (if known) or add a constant of integration

Problems

- Q1 motion of a particle is given by $x = t^4 - 3t^2 - t$ where x is in meter, t in seconds. Find position, velocity, acceleration at t = 3 seconds.

Steps are:

1. Differentiate the given displacement equation find velocity
2. Differentiate the velocity equation and find acceleration

Answer: (x = 51meter, v = 89 m/sec., a = 102m/sec. square)

Problems

Q2 the motion of particle is governed by $a = t^3 - 2t^2 + 7$. It moves in straight line at t=1 second, v=3.5 m/sec. and x = 9.30 m. Find displacement, velocity, acceleration when t = 2 seconds.

Steps are:

1. $a = t^3 - 2t^2 + 7 = dv/dt$ hence $dv = (t^3 - 2t^2 + 7)\, dt$
2. Integrate it find equation for v and value of C_1
3. Now $v = dx/dt$ hence $dx = (t^4/4 - 2t^3/3 + 7t - 3\,)dt$
4. Integrate it and find equation for x and value of C_2
5. Answer x =15.93m, v = 9.67m/sec., a = 7m/sec. square,
6. Displacement = (15.93-9.00) = 6.93m

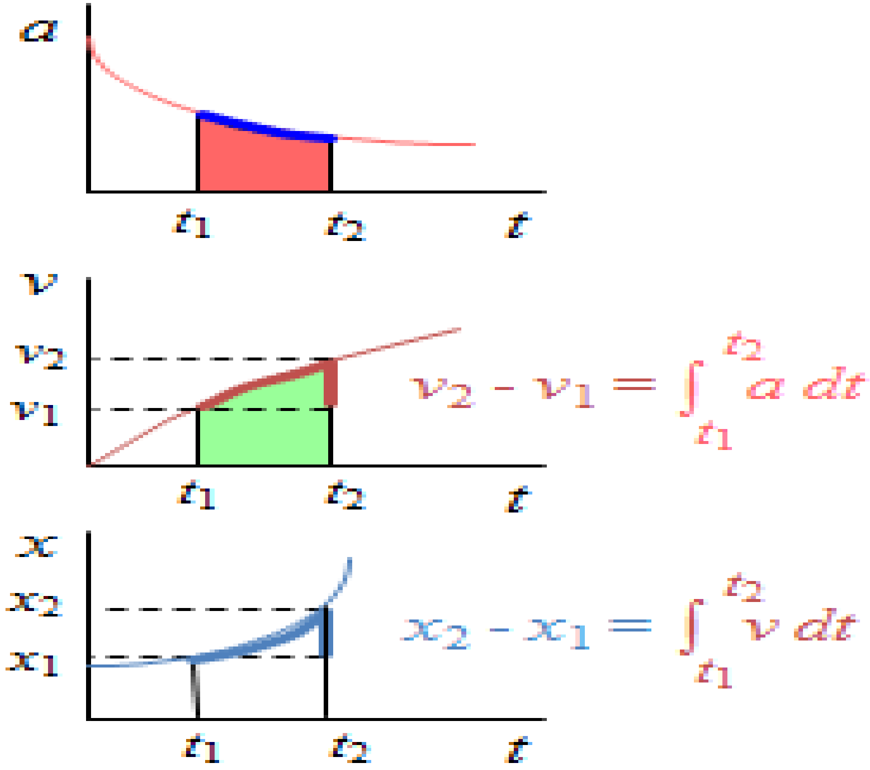
- When particle's motion is erratic, it is best described graphically using a series of curves that can be generated experimentally from computer output.

- A graph can be established describing the relationship with any two of the variables, a, v, s, t

- using the kinematics equations $a = dv/dt$, $v = ds/dt$, $a\, ds = v\, dv$

Motion Diagrams

Sometimes it is convenient to use a *graphical solution* for problems involving rectilinear motion of a particle. The graphical solution most commonly involves $x - t$, $v - t$, and $a - t$ curves.



At any given time t,

v = slope of $x - t$ curve

a = slope of $v - t$ curve

while over any given time interval t_1 to t_2,

$v_2 - v_1$ = area under $a - t$ curve

$x_2 - x_1$ = area under $v - t$ curve

Displacement – Time diagram

Given the *s-t Graph, construct the v-t Graph*

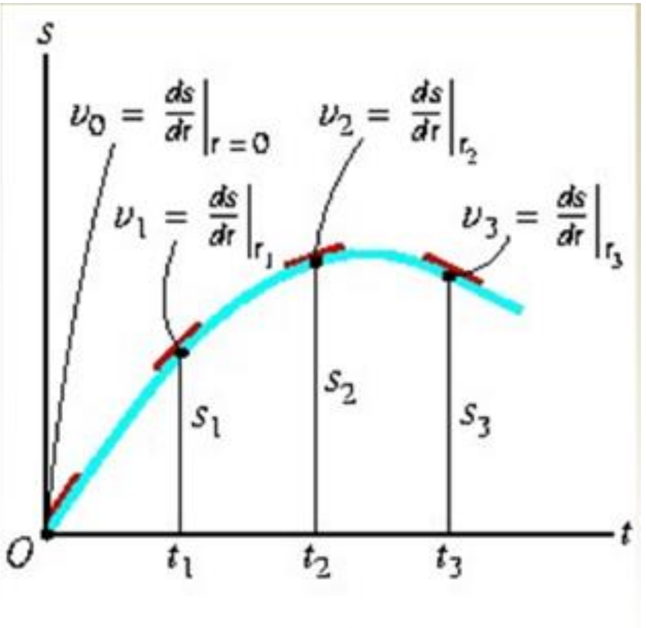
•The *s-t graph can be plotted if the position of the* particle can be *determined experimentally during a* period of time *t.*

•To determine the particle's velocity as a function of time, the *v-t Graph, use v = ds/dt*

•Velocity as any instant is determined by measuring the slope of the *s-t graph*

When displacement of particle is maximum or minimum velocity of particle is zero.

$$\frac{ds}{dt} = v$$



(a)

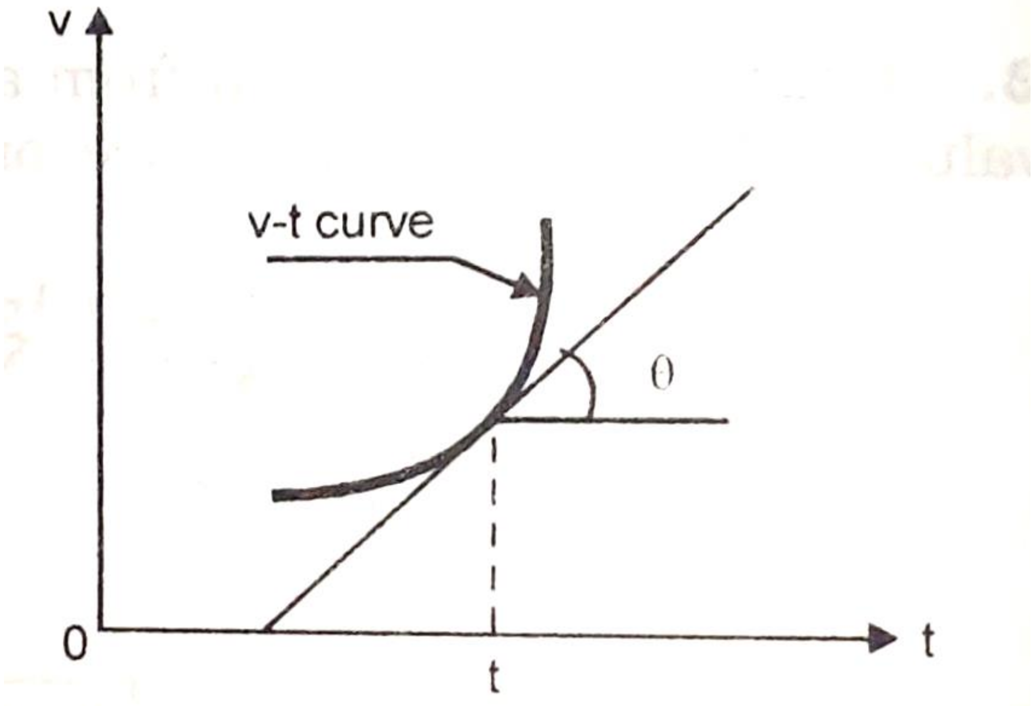
Slope of *s-t graph=velocity*

Velocity Time Diagram

- This is drawn with velocity on y axis and time on x axis.
- As a = dv/dt , slope of v-t curve gives acceleration of particle at that instant.
- Now v = dx/dt

so   dx = vdt

When velocity of particle is maximum or minimum acceleration of particle is zero.

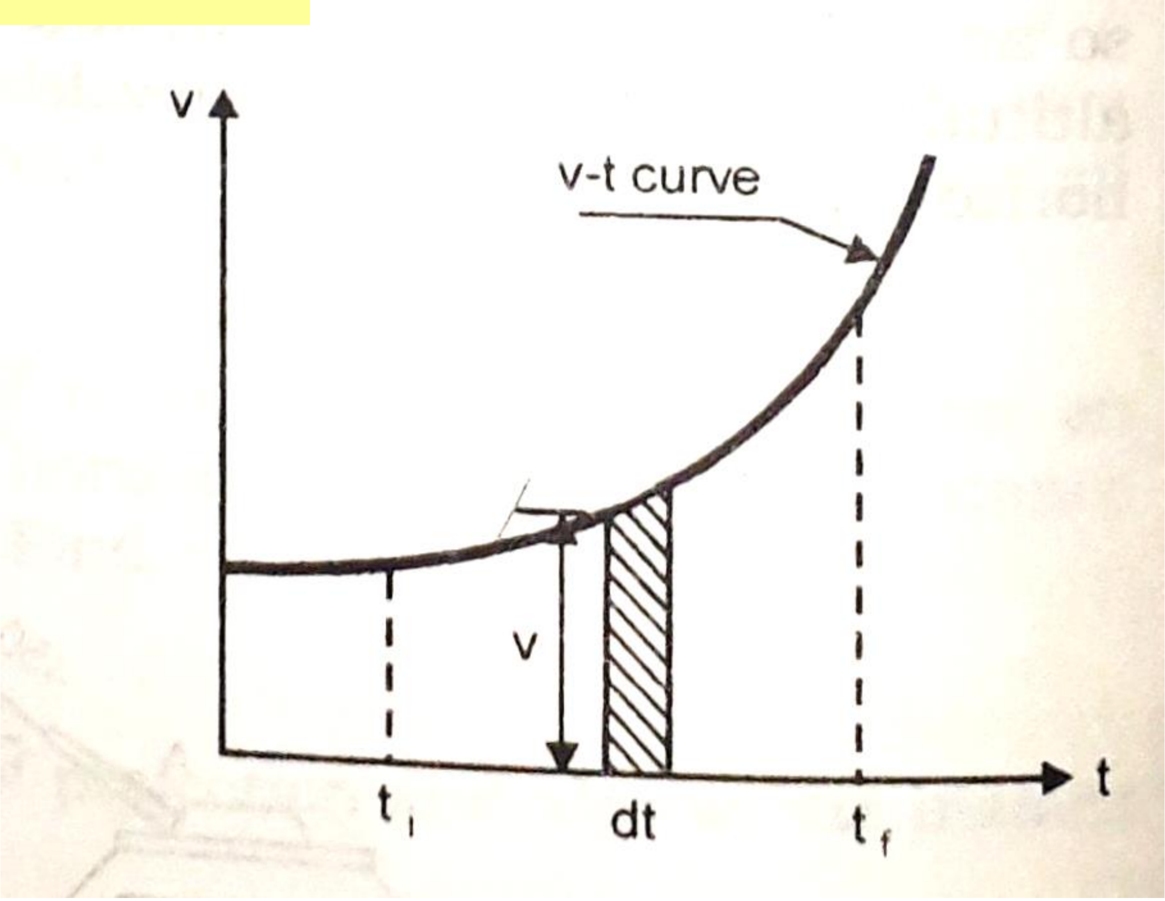


Velocity Time Diagram

$$v = \frac{dx}{dt} \quad \Rightarrow dx = vdt \quad \Rightarrow \int_{x_0}^{x} dx = \int_{0}^{t} vdt \quad \Rightarrow x - x_0 = \int_{0}^{t} vdt$$

Or

$x-x_0$ = area under v-t curve

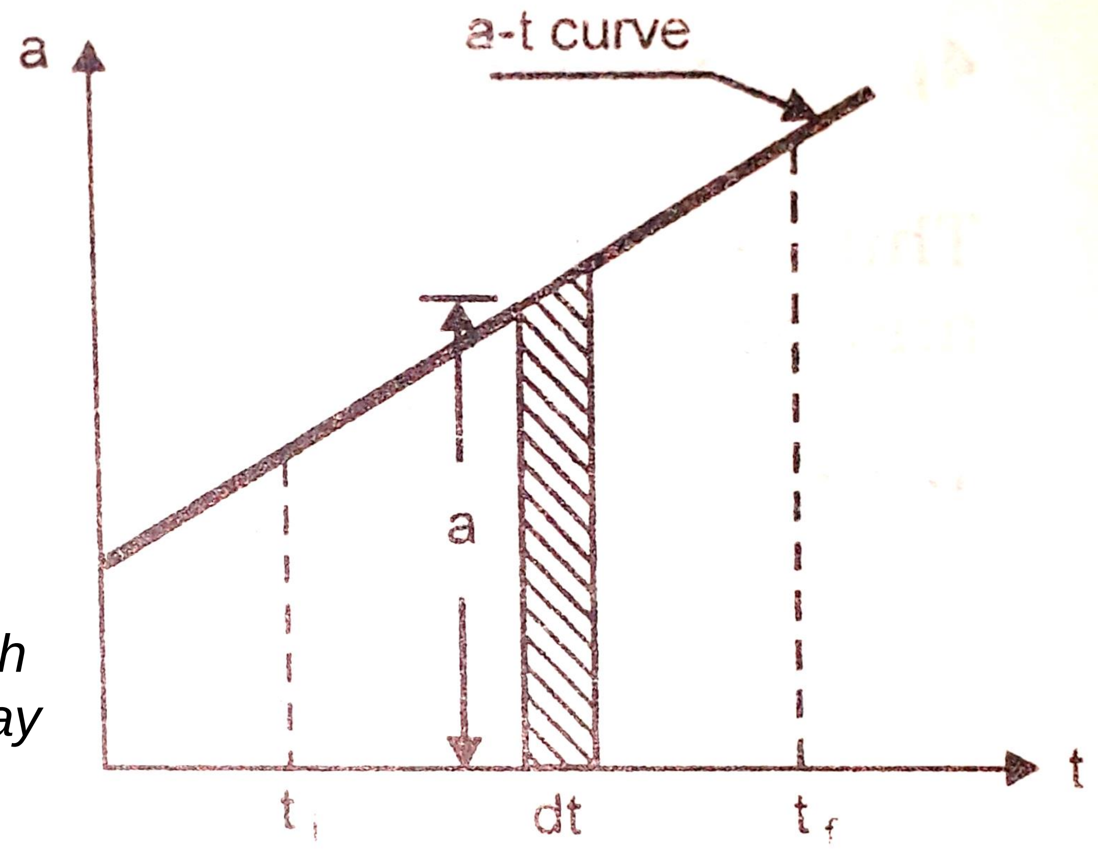


Acceleration Time Diagram

- Given the *a-t Graph, construct the v-t Graph*
- • When the *a-t graph is known, the v-t graph may* be constructed using *a = dv/dt*

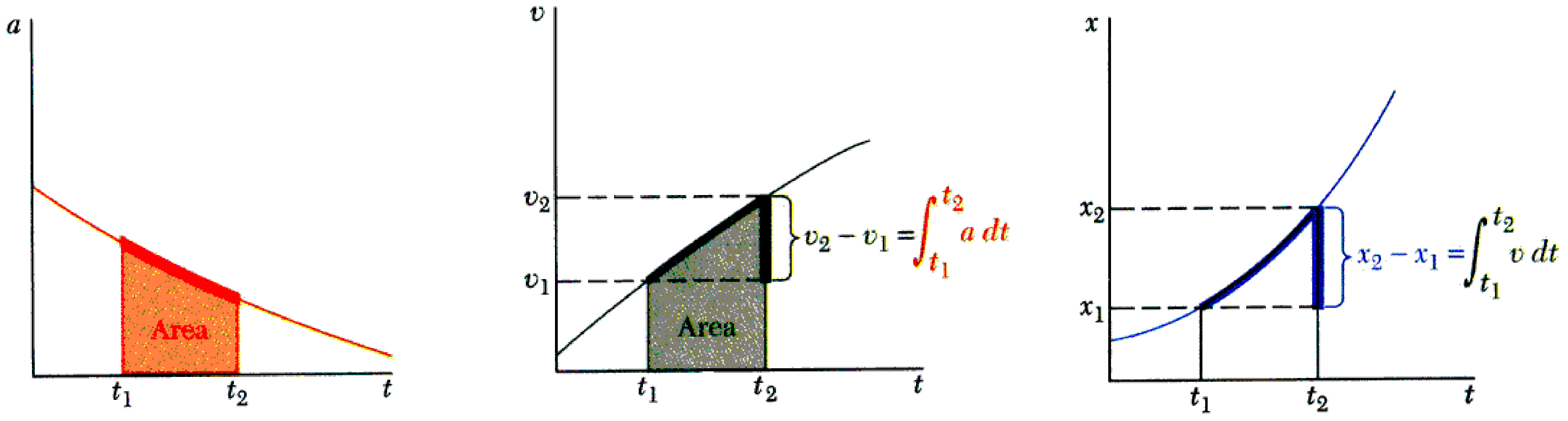
$$\Delta v = \int a\,dt$$

Change in velocity = Area under *a-t* graph



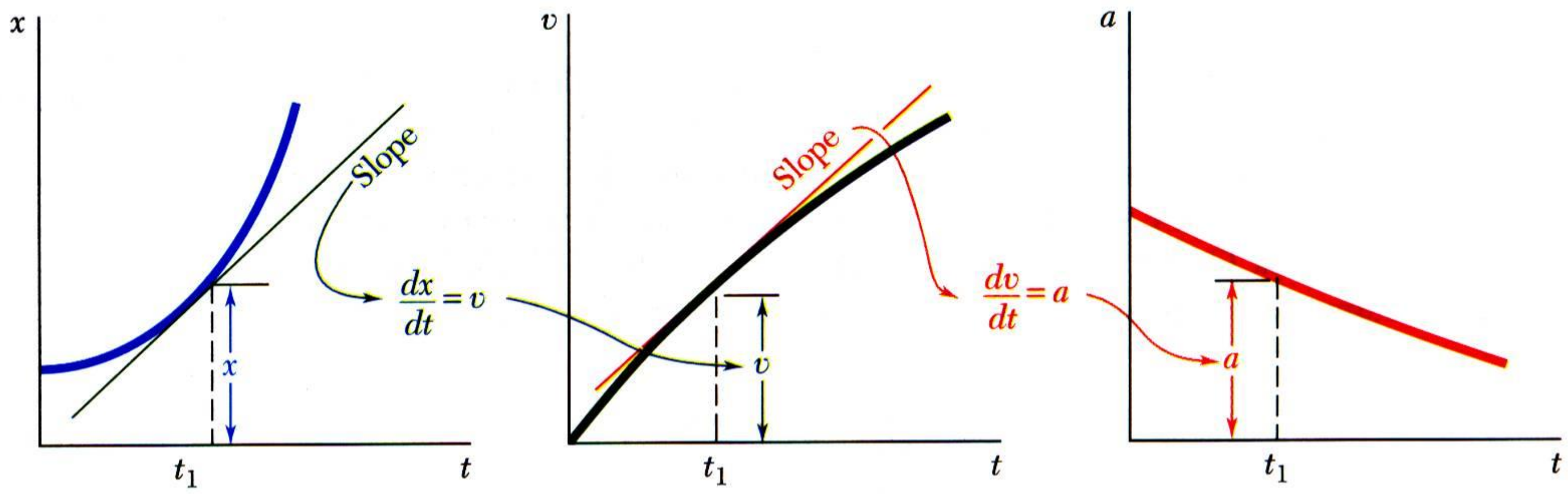
• Knowing particle's initial velocity *v0, and add to this small increments* of area *(Δv)*
• Successive points *v1 = v0 + Δv, for* the *v-t graph*
• Each eqn. for each segment of the *a-t graph may be integrated to yield* eqns. for corresponding segments of the *v-t graph*

Graphical Solution of Rectilinear-Motion Problems



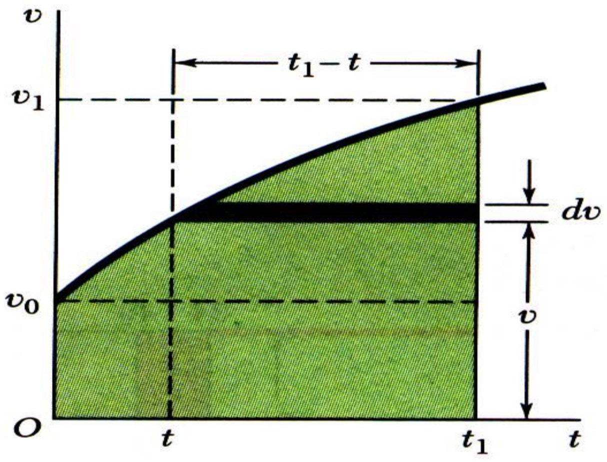
- Given the *a*-*t* curve, the change in velocity between t_1 and t_2 is equal to the area under the *a*-*t* curve between t_1 and t_2.
- Given the *v*-*t* curve, the change in position between t_1 and t_2 is equal to the area under the *v*-*t* curve between t_1 and t_2.

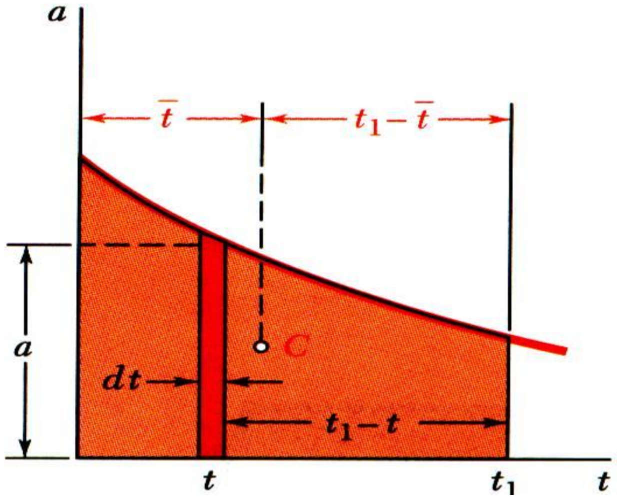
Graphical Solution of Rectilinear-Motion Problems



- Given the *x*-*t* curve, the *v*-*t* curve is equal to the slope of *x*-*t* curve
- Given the *v*-*t* curve, the *a*-*t* curve is equal to the slope *v*-*t* curve

Other Graphical Methods





- *Moment-area method* to determine particle position at time *t* directly from the *a-t* curve:

$$x_1 - x_0 = \text{area under } v - t \text{ curve}$$

$$= v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) dv$$

using $dv = a\,dt$,

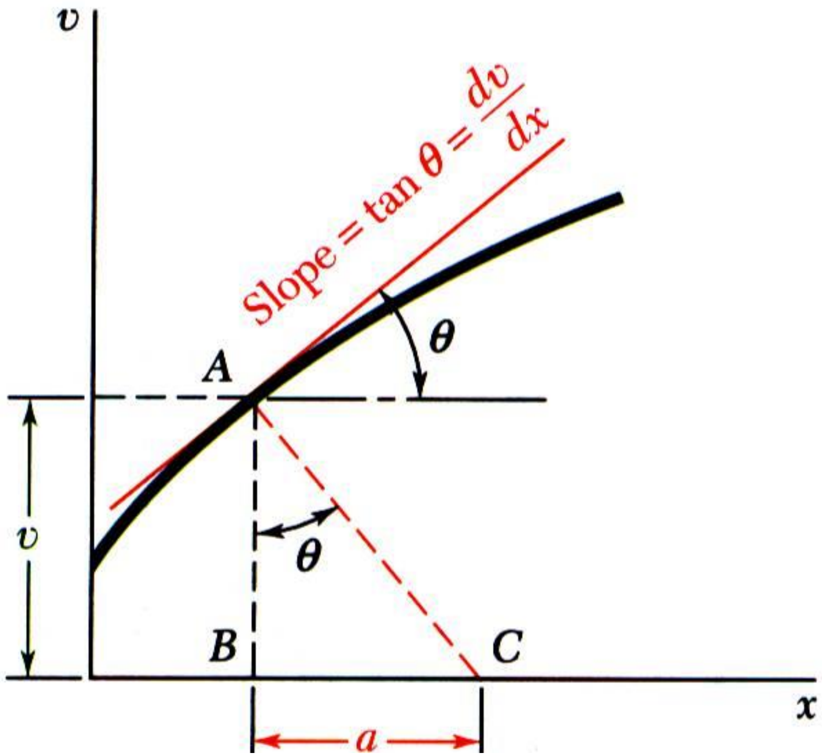
$$x_1 - x_0 = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) a\,dt$$

$$\int_{v_0}^{v_1} (t_1 - t) a\,dt = \text{first moment of area under } a\text{-}t \text{ curve with respect to } t = t_1 \text{ line.}$$

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{-}t \text{ curve})(t_1 - \bar{t})$$

$$\bar{t} = \text{abscissa of centroid } C$$

Other Graphical Methods



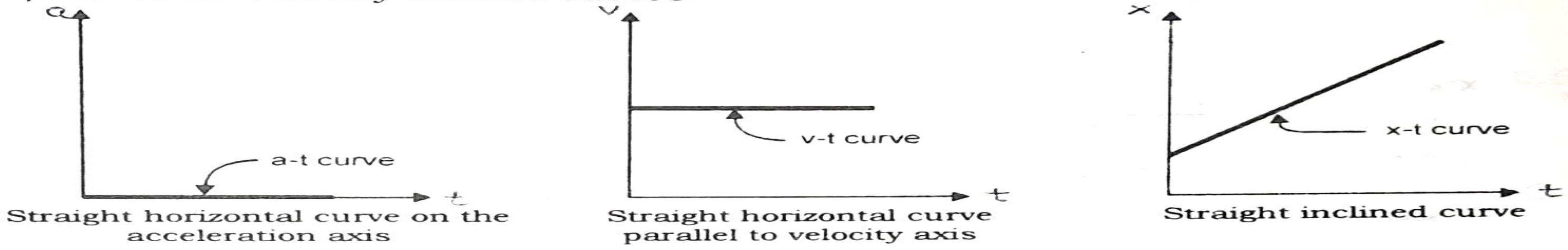
- Method to determine particle acceleration from *v*-*x* curve:

$$a = v\frac{dv}{dx}$$

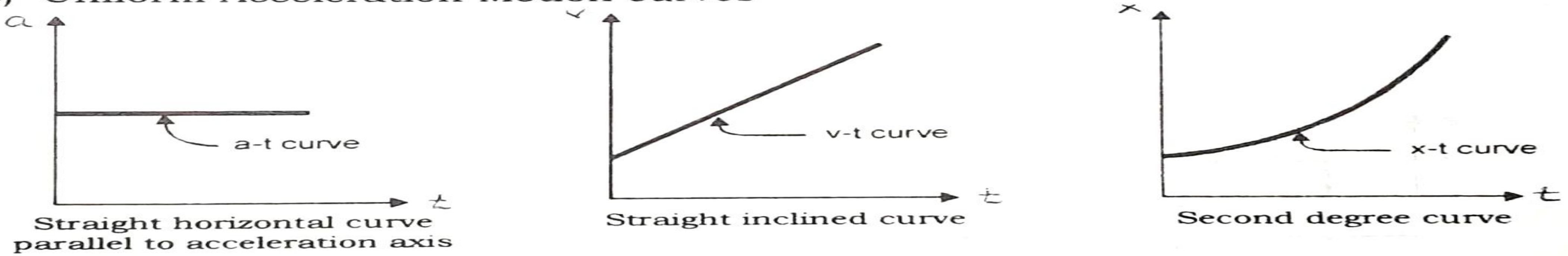
$$= AB\tan\theta$$

$$= BC = \textit{ subnormal} \text{ to } v\text{-}x \text{ curve}$$

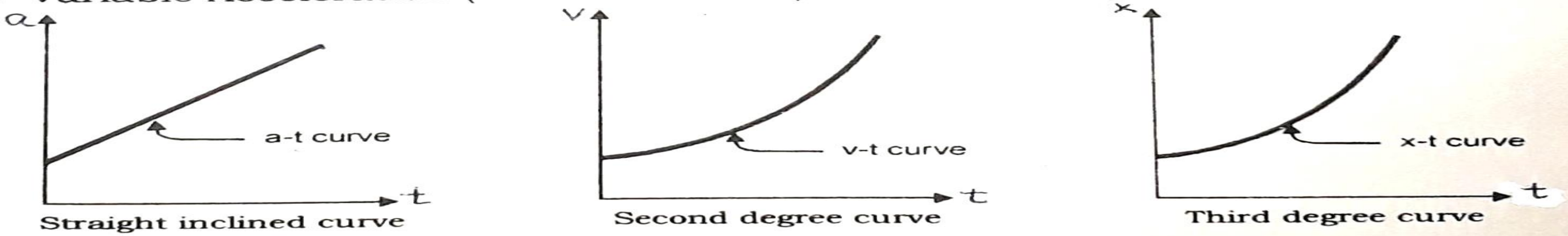
a) Uniform Velocity Motion curves

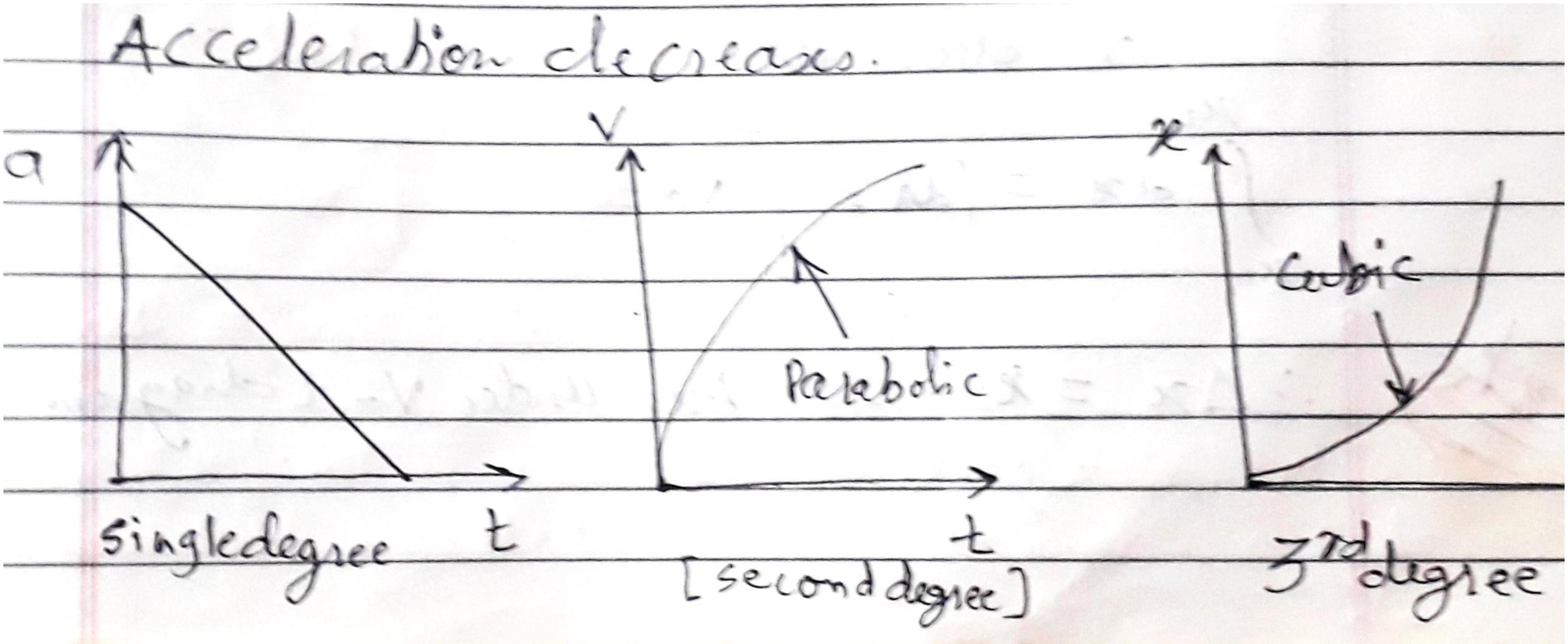


b) Uniform Acceleration Motion curves

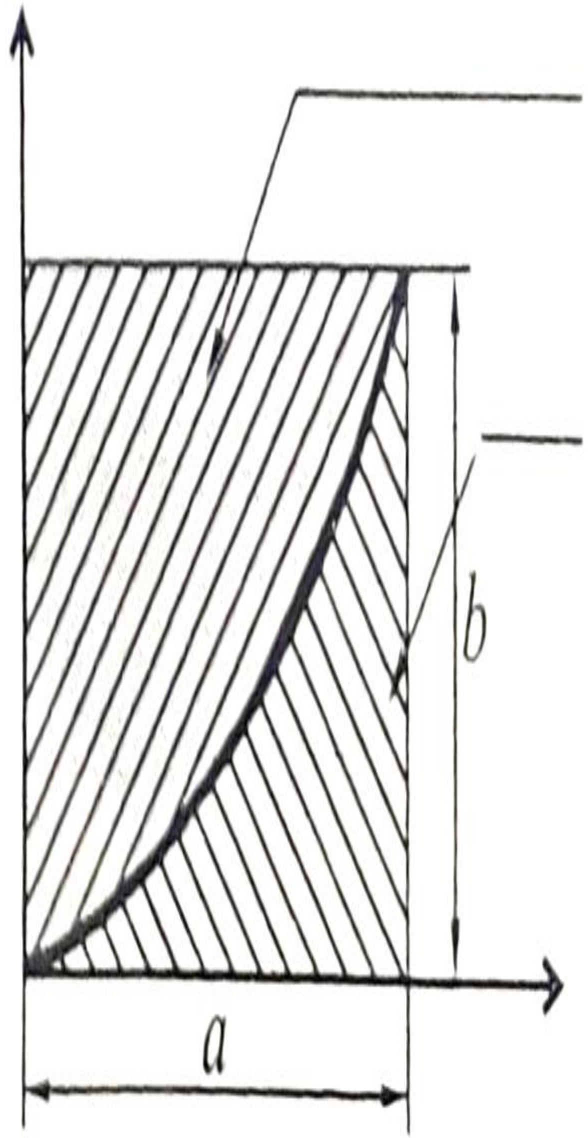


c) Variable Acceleration (Linear Variation) Motion curves




Acceleration decreases.
a
t
Single degree
v
Parabolic
t
[second degree]
x
Cubic
3rd degree

Area bounded by curve



$$A_1 = \frac{nab}{n+1}$$

$$A_2 = \frac{ab}{n+1}$$

$$A = A_1 + A_2 = ab$$

n = Curve of degree of polynomial

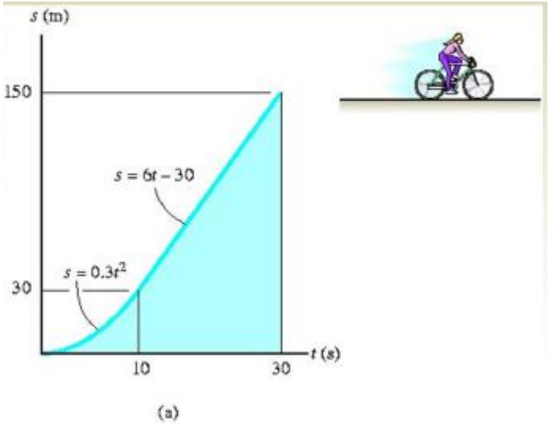
Important points to remember

- If a-t curve is horizontal line (zero degree) then v-t curve is inclined line (single degree) and x-t curve is parabolic curve (second degree)
- Slope of motion curve increases from a-t curve towards v-t curve.

Problems

- Q1 A bicycle moves along a straight road such that it position is

described by the graph as shown. Construct the *v-t and a-t graphs for 0 ≤ t ≤ 30s.*

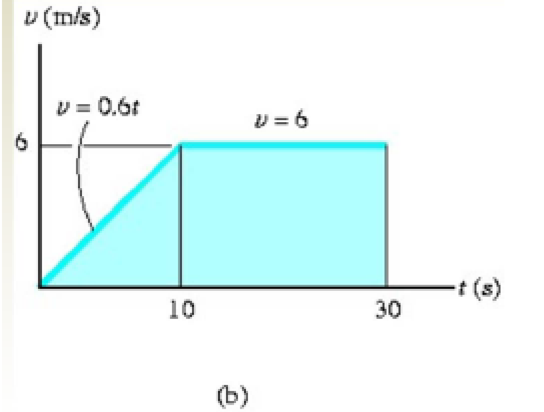


v-t Graph. The v-t graph can be determined by differentiating the eqns. defining the *s-t graph*

$$0 \le t \le 10s; \quad s = 0.3t^2 \quad v = \frac{ds}{dt} = 0.6t$$

$$10s \le t \le 30s; \quad s = 6t - 30 \quad v = \frac{ds}{dt} = 6$$

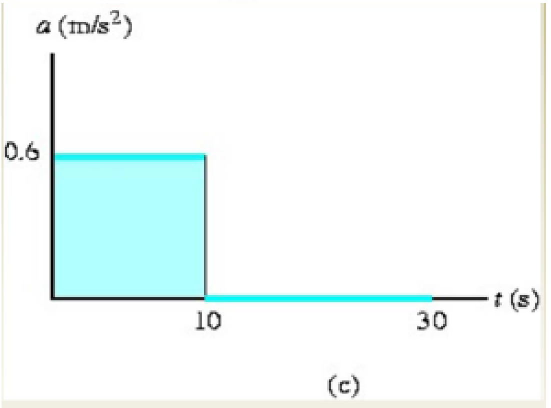
$$v = \frac{\Delta s}{\Delta t} = \frac{150 - 30}{30 - 10} = 6m/s$$



a-t Graph. The a-t graph can be determined by differentiating the eqns. defining the lines of the *v-t graph.*

$$0 \le t \le 10s; \quad v = 0.6t \quad a = \frac{dv}{dt} = 0.6$$

$$10 < t \le 30s; \quad v = 6 \quad a = \frac{dv}{dt} = 0$$



Q 2 A test car starts from rest and travels along a straight track such that it accelerates at a constant rate for 10 s and then decelerates at a constant rate. Draw the v-t and s-t graphs and determine the time t’ needed to stop the car. How far has the car traveled?

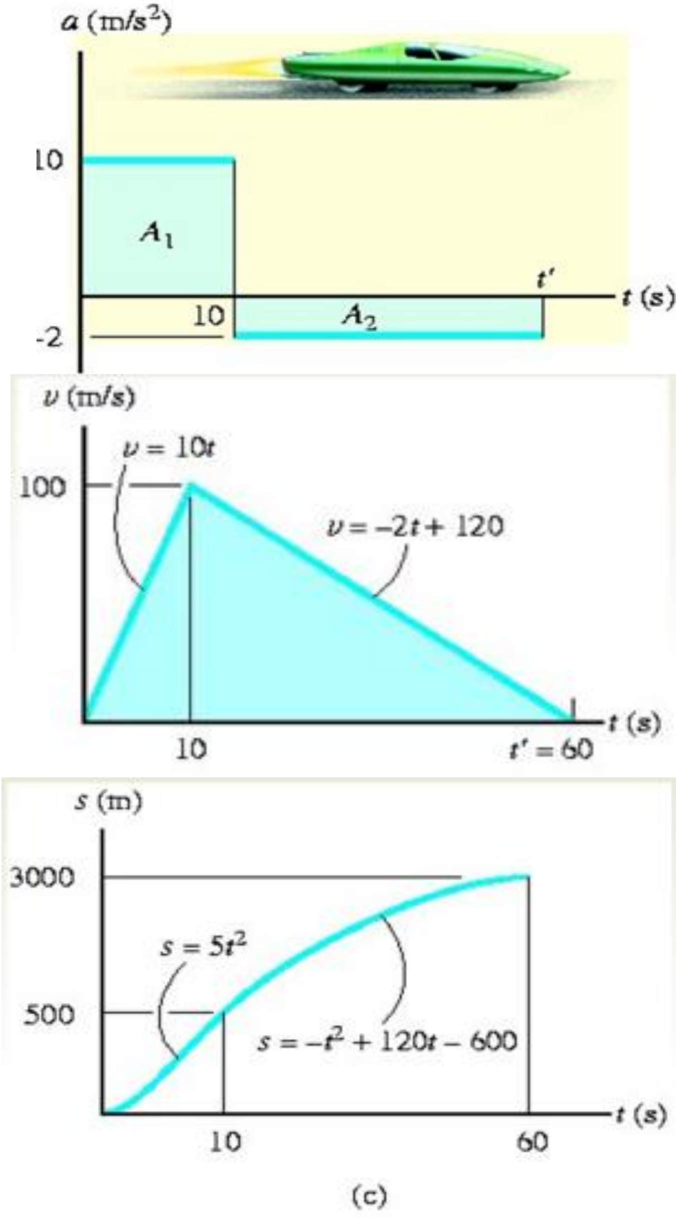
v-t Graph. The v-t graph can be determined by integrating the straight-line segments of the *a-t* graph. Using *initial condition v = 0 when t = 0,*

$$0 \le t \le 10s \quad a = 10; \quad \int_0^v dv = \int_0^t 10\,dt, \quad v = 10t$$

When *t = 10s, v = 100m/s, using this as initial* condition for the next time period, we have

$$10s \le t \le t'; \quad a = -2; \quad \int_{100}^{v} dv = \int_{10}^{t} -2\,dt, \quad v = -2t + 120$$

When *t = t’ we require v = 0. This yield t’ = 60 s*



s-t Graph. *Integrating the eqns. of the v-t graph* yields the corresponding eqns. of the *s-t graph*. Using the *initial conditions s = 0 when t = 0,*

$$0 \le t \le 10s; \quad v = 10t; \quad \int_0^s ds = \int_0^t 10t\, dt, \quad s = 5t^2$$

When t = 10s, s = 500m. Using this initial condition,

$$10s \le t \le 60s; \quad v = -2t + 120; \quad \int_{500}^s ds = \int_{10}^t (-2t + 120) dt$$

$$s = -t^2 + 120t - 600$$

When t' = 60s, the position is s = 3000m

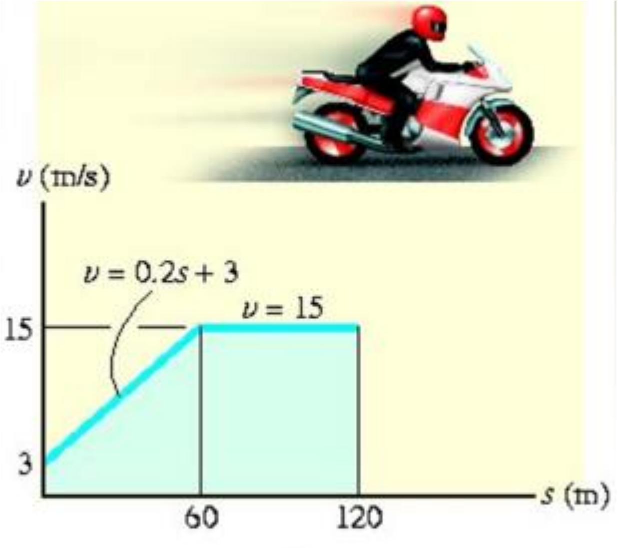
The *v-s graph describing the motion of a motorcycle* is as shown. Construct the *a-s graph of* the motion and determine the time needed for the motorcycle to reach the position *s = 120 m.*

$$0 \le s \le 60m; \quad v = 0.2s + 3$$

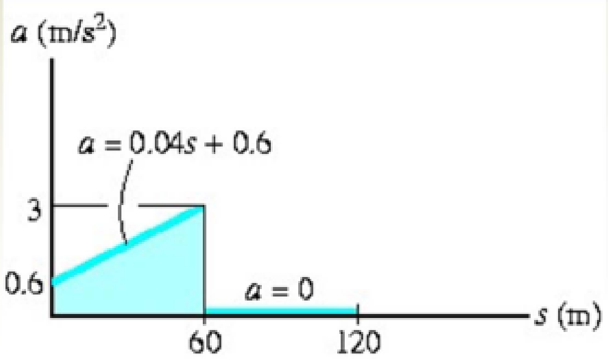
$$a = v\frac{dv}{ds} = 0.04s + 0.6$$

$$60m < s \le 120m; \quad v = 15;$$

$$a = v\frac{dv}{ds} = 0$$



(a)

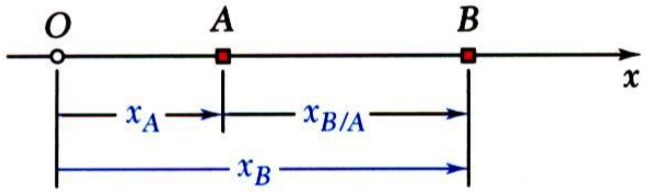


(b)

Ans: At s=60m, t=8.05 s

Ans: At s=120m, t=12.0 s

Motion of Several Particles: Relative Motion



- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

$$x_{B/A} = x_B - x_A = $$ relative position of *B* with respect to *A*

$$x_B = x_A + x_{B/A}$$

$$v_{B/A} = v_B - v_A = $$ relative velocity of *B* with respect to *A*

$$v_B = v_A + v_{B/A}$$

$$a_{B/A} = a_B - a_A = $$ relative acceleration of *B* with respect to *A*

$$a_B = a_A + a_{B/A}$$

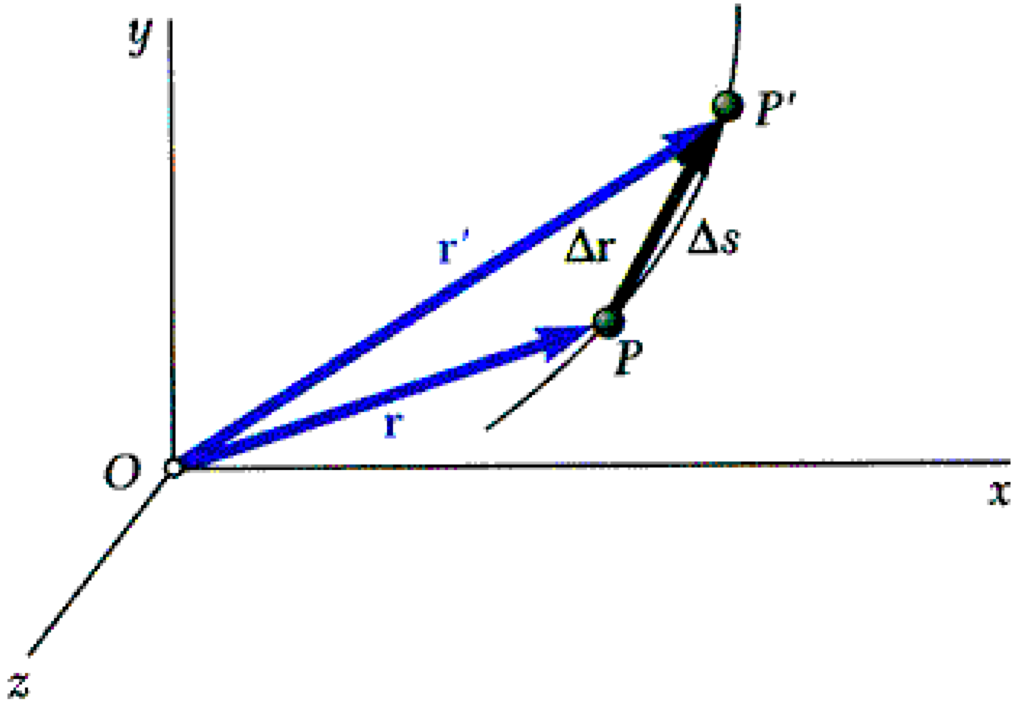
Curvilinear Motion: Position, Velocity & Acceleration

The softball and the car both undergo curvilinear motion.

- A particle moving along a curve other than a straight line is in *curvilinear motion*.

Curvilinear Motion: Position, Velocity & Acceleration

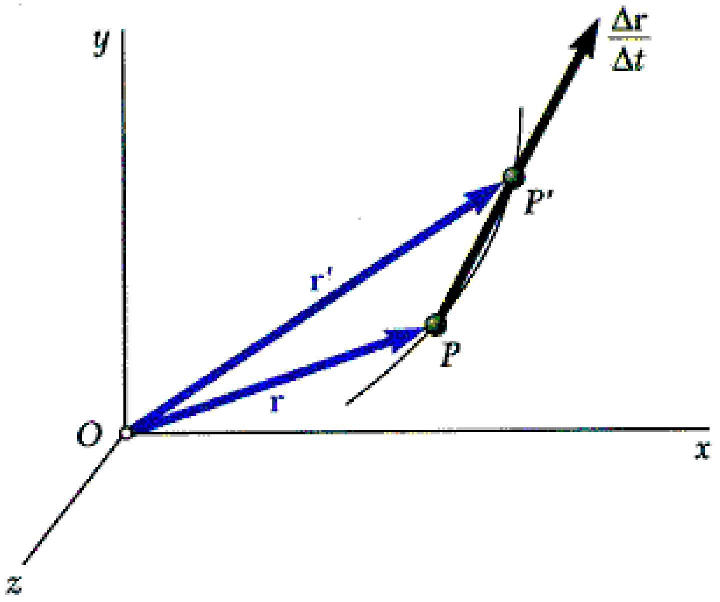
- The *position vector* of a particle at time *t* is defined by a vector between origin *O* of a fixed reference frame and the position occupied by particle.
- Consider a particle which occupies position *P* defined by r at time *t* and *P'* defined by r' at *t* + *Dt*,



Curvilinear Motion: Position, Velocity & Acceleration

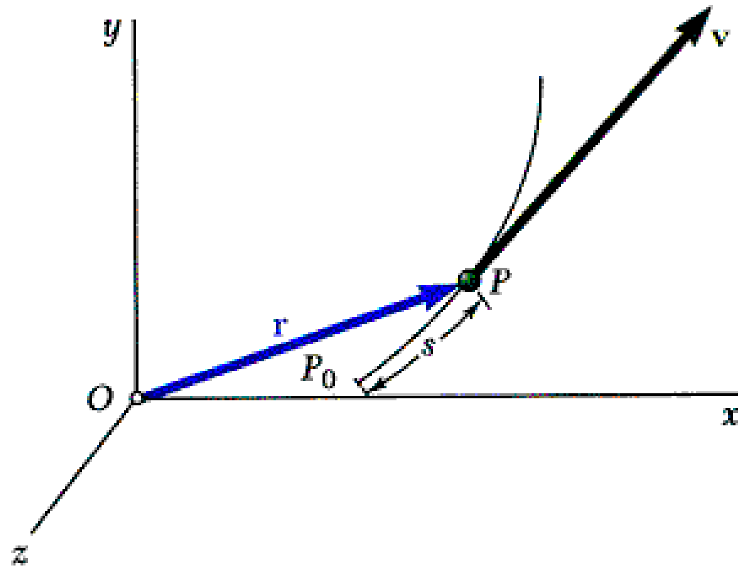
Instantaneous velocity
(vector)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



Instantaneous speed
(scalar)

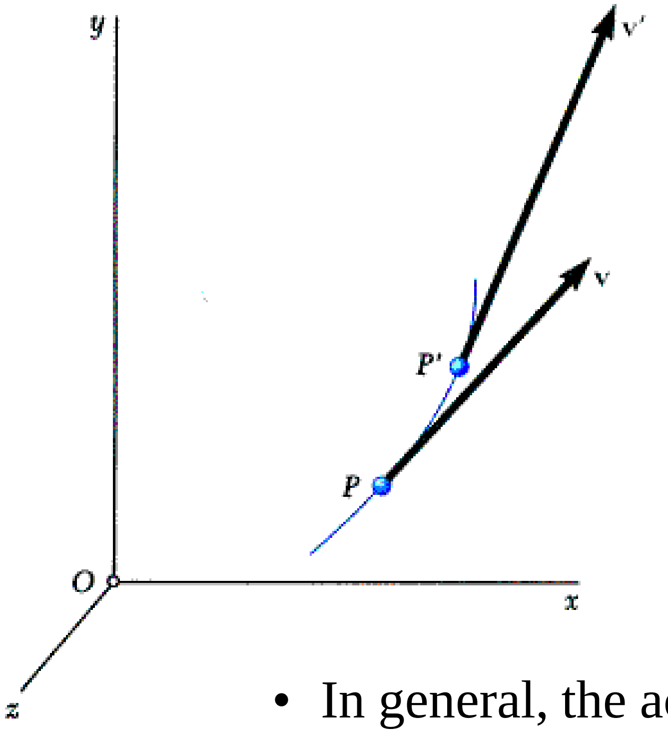
$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

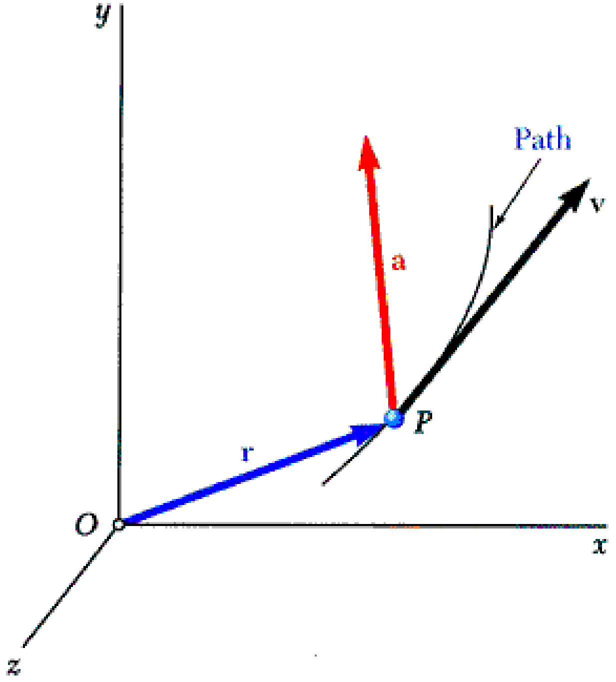


Curvilinear Motion: Position, Velocity & Acceleration

- Consider velocity $\vec{v}$ of a particle at time *t* and velocity $\vec{v}'$ at $t + \Delta t$,

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} =$$ instantaneous acceleration (vector)





- In general, the acceleration vector is not tangent to the particle path and velocity vector.