TRIPLE INTEGRATION

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EVALUATION OF TRIPLE INTEGRAL:

For the purpose of evaluation, it can be expressed as the repeated integral

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz$$

The order of integration depends upon the limits.

Let z_1 *and* z_2 be functions of x, y. i.e. $z_1 = f_1(x, y), z_2 = f_2(x, y)$, let y_1 *and* y_2 be functions of x, i.e. $y_1 = \emptyset_1(x), y_2 = \emptyset_2(x)$ and x_1 and x_2 be constants i.e. $x_1 = a, x_2 = b$ then the integral I is evaluated as follows:

$$I = \int_{x_1=a}^{x_2=b} \int_{y_1=\emptyset_1(x)}^{y_2=\emptyset_2(x)} \int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x, y, z)\, dx\, dy\, dz$$

TYPE I : WHEN THE LIMITS OF INTEGRATION ARE GIVEN

Evaluate the following integrals.

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}}$

we first integrate wrt z, then wrt y and then wrt x

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}} \right] dy\, dx$$

$1-x^2-y^2 = a^2$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_0^a \frac{dz}{\sqrt{a^2-z^2}} \right] dy\, dx$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \left(\sin^{-1}\left(\frac{z}{a}\right)\right)_0^a dy\, dx$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \frac{\pi}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$$

$$= \frac{\pi}{2}\int_0^1 (y)_0^{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{2}\left[0 + \frac{1}{2}\frac{\pi}{2} - 0\right] = \frac{\pi^2}{8}$$

2. $\int_0^2 \int_0^x \int_0^{2x+2y} e^{x+y+z}\,dz\,dy\,dx$

First we integrate wrt z, then wrt y and then wrt x

$$I = \int_0^2\int_0^x\int_0^{2x+2y} e^{x+y}\cdot e^{z}\,dz\,dy\,dx$$

$$= \int_0^2\int_0^x e^{x+y}\cdot\left(e^{z}\right)_0^{2x+2y}\,dy\,dx$$

$$= \int_0^2\int_0^x e^{x+y}\cdot\left[e^{2x+2y} - 1\right]dy\,dx$$

$$= \int_0^2\int_0^x e^{3x+3y} - e^{x+y}\,dy\,dx$$

$$= \int_0^2\left(\frac{e^{3x+3y}}{3} - e^{x+y}\right)_0^x dx$$

$$= \int_0^2\left(\frac{e^{6x}}{} - e^{2x} - \frac{e^{3x}}{3} + e^{x}\right)dx$$

$$= \int_0^2 \left(\frac{e^{6x}}{3} - e^{2x} - \frac{e^{3x}}{3} + e^{x} \right) dx$$

$$= \left(\frac{e^{6x}}{18} - \frac{e^{2x}}{2} - \frac{e^{3x}}{9} + e^{x} \right)_0^2$$

$$= \frac{e^{12}}{18} - \frac{e^{4}}{2} - \frac{e^{6}}{9} + e^{2} - \left(\frac{1}{18} - \frac{1}{2} - \frac{1}{9} + 1 \right)$$

$$= \frac{e^{12}}{18} - \frac{e^{4}}{2} - \frac{e^{6}}{9} + e^{2} - \frac{4}{9}$$

3. $\int_0^{log\,2} \int_0^{x} \int_0^{x+y} e^{x+y+z} dx\, dy\, dz.$ **(HW)**

4. $\int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x+y+z) dz\, dx\, dy.$

5/24/2021 11:30 AM

5. $\int_0^{\pi} 2d\theta \int_0^{a(1+cos\theta)} r\,dr \int_0^h \left[1 - \frac{r}{a(1+cos\theta)}\right] dz$

Sol^n:

$$I = \int_0^{\pi} 2\,d\theta \int_0^{a(1+\cos\theta)} r\,dr \int_0^h \left[1 - \frac{r}{a(1+\cos\theta)}\right] dz$$

$$= \int_0^{\pi} 2\,d\theta \int_0^{a(1+\cos\theta)} r\,dr \cdot \left[1 - \frac{r}{a(1+\cos\theta)}\right] (z)_0^h$$

$$= 2h \int_0^{\pi} d\theta \int_0^{a(1+\cos\theta)} \left[r - \frac{r^2}{a(1+\cos\theta)}\right] dr$$

$$= 2h \int_0^{\pi} d\theta \left[\frac{r^2}{2} - \frac{r^3}{3a(1+\cos\theta)}\right]_0^{a(1+\cos\theta)}$$

$$= 2h \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2} - \frac{a^3(1+\cos\theta)^3}{3a(1+\cos\theta)}\, d\theta$$

$$= \frac{a^2h}{3} \int_0^{\pi} (1+\cos\theta)^2 d\theta$$

$$= a^2h \int^{\pi} (1 + 2\cos\theta + \cos^2\theta) \ldots$$

$$= \frac{a^2h}{3}\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \frac{a^2h}{3}\int_0^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2h}{3}\left[\frac{3}{2}(\theta)+2\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$I = \frac{a^2h}{3}\cdot\frac{3}{2}\pi = \frac{\pi a^2h}{2}$$

6. $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{(a^2-r^2)/a} r\,d\theta dr\,dz$ **(HW)**

7. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx$

Soln:- First we integrate wrt z, then y and then x

$$I = \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}(z)_{x^2+3y^2}^{8-x^2-y^2}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\left[8-2x^2-4y^2\right]dy\,dx$$

$$= \int_{-2} \int_{-\sqrt{4-x^2}/2} \left[8 - 2x^2 - 4y^2 \right] dy\, dx$$

$$= \int_{-2}^{2} (8-2x^2)\,(y)\Big|_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} - \frac{4}{3}\left(y^3\right)\Big|_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} dx$$

$$= \int_{-2}^{2} 2(4-x^2)\left[\sqrt{4-x^2}\right] - \frac{4}{3}\left(\frac{(4-x^2)^{3/2}}{8}\right) dx$$

$$= \int_{-2}^{2} \frac{11}{6}\,(4-x^2)^{3/2}\, dx$$

put $x = 2\sin\theta \quad dx = 2\cos\theta\, d\theta$

x	-2	2
$\theta =$	$-\frac{\pi}{2}$	$\pi/2$

$$I = \int_{-\pi/2}^{\pi/2} \frac{11}{6} \cdot 4\,(1-\sin^2\theta)^{3/2} \cdot 2\cos\theta\, d\theta$$

$$= \frac{88}{6} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{88}{3} \int_{0}^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{88}{3} \cdot \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{44}{3} \cdot \frac{\Gamma 5/2\, \Gamma 1/2}{\Gamma 3}$$

$$= \frac{44}{3} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2}\, \Gamma\frac{1}{2}\, \Gamma\frac{1}{2}}{2} = \frac{11}{2}\pi$$

8. $\int_0^a \int_0^a \int_0^a (yz + zx + xy)dx\, dy\, dz.$

$$I = \int_0^a \int_0^a yz(x)_0^a + z\left(\frac{x^2}{2}\right)_0^a + \left(\frac{x^2}{2}\right)_0^a y \; dy dz$$

$$= \int_0^a \int_0^a \left(ayz + \frac{a^2}{2}z + \frac{a^2}{2}y\right) dy dz$$

$$= \int_0^a az\left(\frac{y^2}{2}\right)_0^a + \frac{a^2}{2}z(y)_0^a + \frac{a^2}{2}\left(\frac{y^2}{2}\right)_0^a dz$$

$$= \int_0^a \frac{a^3}{2}z + \frac{a^3}{2}z + \frac{a^4}{4} \; dz$$

$$= \frac{a^3}{2}\left(\frac{z^2}{2}\right)_0^a + \frac{a^3}{2}\left(\frac{z^2}{2}\right)_0^a + \frac{a^4}{4}(z)_0^a$$

$$= \frac{a^5}{4} + \frac{a^5}{4} + \frac{a^5}{4}$$

$$= \frac{3}{4}a^5$$

9. $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dx\, dy\, dz$ **(HW)** $\rightarrow$ order z, y, x

10. $\int_0^2 \int_1^2 \int_0^{yz} xyz\, dx\, dy\, dz$ **(HW)** $\rightarrow$ order x, y, z

11. $\int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\, dy\, dz}{(1+x^2+y^2+z^2)^2}$

we transform the integral from cartesian to spherical polar coordinates because of the term $x^2+y^2+z^2$

we put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$ and $dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

$z = r\cos\theta$

Now $x^2+y^2+z^2 = r^2$

Since x, y, z all vary from 0 to ∞, the region of integration is the first octant in which θ, ϕ vary from 0 to $\frac{\pi}{2}$ and r varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{0}^{\infty} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(1+r^2)^2}$$

$$= \left(\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\right)\left(\int_{0}^{\pi/2}d\phi\right)\left(\int_{0}^{\infty}\frac{r^2}{(1+r^2)^2}\,dr\right)$$

put $r = \tan t$

$dr = \sec^2 t\,dt$

when $r=0,\ t=0$

$r=\infty,\ t=\frac{\pi}{2}$

$$= \left[-\cos\theta\right]_0^{\pi/2}\left[\phi\right]_0^{\pi/2}\int_0^{\pi/2}\frac{\tan^2 t}{\sec^4 t}\sec^2 t\,dt$$

$$= 1\cdot\frac{\pi}{2}\int_0^{\pi/2}\sin^2 t\,dt = \frac{\pi}{2}\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right)$$

$$I = \frac{\pi}{4}\cdot\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} = \frac{\pi^2}{8}$$

12. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} (x^2+y^2+z^2)\,dx\,dy\,dz$

$z = 0$ to $\sqrt{a^2-x^2-y^2} \rightarrow x^2+y^2+z^2 = a^2$
sphere radius a

The first octant

Convert into spherical coordinates.

$x = r\sin\theta\cos\phi$, $y = r\sin\phi\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

Since this the first octants

θ and ϕ will vary from 0 to $\frac{\pi}{2}$

r will vary from 0 to a

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{0}^{a} r^2(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

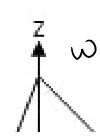
$$= \left(\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta\right)\left(\int_{\phi=0}^{\pi/2}d\phi\right)\left(\int_0^a r^4\,dr\right)$$

$$I = \frac{\pi a^5}{10}$$

TYPE II : WHEN THE REGION OF INTEGRATION IS BOUNDED BY PLANES

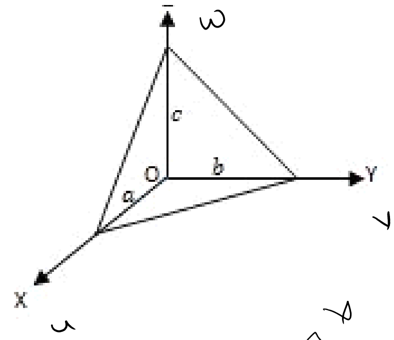
1. Evaluate $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by the planes
$x = 0, y = 0, z = 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

Soln:
$x = 0 \longrightarrow yz$ plane
$y = 0 \longrightarrow xz$ plane
$z = 0 \longrightarrow xy$ plane



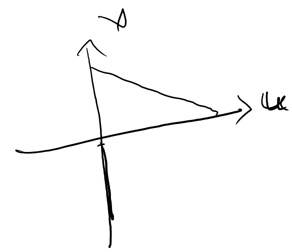
we substitute $x = au$, $y = bv$,

we substitute $x = au$, $y = bv$,
$z = cw$



$$dx\,dy\,dz = abc\,du\,dv\,dw$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow u + v + w = 1$$



$w = 0$ to $1-u-v$
$v = 0$ to $1-u$
$u = 0$ to 1

$$I = \iiint x^2 yz\,dx\,dy\,dz$$

$$= \int_{u=0}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} a^2u^2\,bv\,cw\;abc\,du\,dv\,dw$$

$$= a^3b^2c^2 \int_{u=0}^{1}\int_{0}^{1-u}\int_{0}^{1-u-v} u^2vw\,du\,dv\,dw$$

$$= a^3b^2c^2 \int_{u=0}^{1}\int_{v=0}^{1-u} u^2v\left(\frac{w^2}{2}\right)_0^{1-u-v} dv\,du$$

$$= \frac{a^3b^2c^2}{2}\int_{u=0}^{1}\int_{v=0}^{1-u} u^2v\,(1-u-v)^2\,dv\,du$$

$$= \frac{a^3b^2c^2}{2}\int_{0}^{1}\int^{1-u} u^2v\left[(1-u)^2 - 2(1-u)v + v^2\right] dv\,du$$

$$= \frac{a^3b^2c^2}{2} \int_0^1 \int_0^{1-u} u^2 v \left[(1-u)^2 - 2(1-u)v + v^2\right] dv\,du$$

$$= \frac{a^3b^2c^2}{2} \int_0^1 u^2 \left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2} \int_0^1 u^2 \left[(1-u)^2\frac{(1-u)^2}{2} - 2(1-u)\frac{(1-u)^3}{3} + \frac{(1-u)^4}{4}\right]$$

$$= \frac{a^3b^2c^2}{2} \int_0^1 \frac{u^2(1-u)^4}{12}\, du$$

$$= \frac{a^3b^2c^2}{24} \int_0^1 u^2(1-u)^4\, du$$

$$= \frac{a^3b^2c^2}{24} \cdot B(3,5) = \frac{a^3b^2c^2}{2520}.$$

2. Evaluate $\iiint dx\,dy\,dz$ over the volume of the tetrahedron bounded by $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (HW)

Similar to previous sum substitution

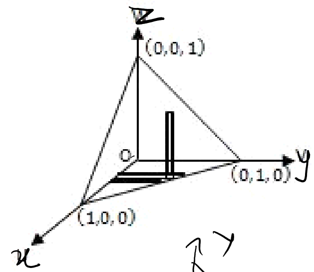
$$I = \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} abc\, dw\,dv\,du = \frac{abc}{6}.$$

5/26/2021 11:00 AM

3. Evaluate $\iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$

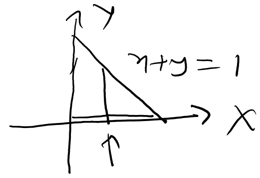
$z \to xy$ plane to $x+y+z=1$

$z=0$ to $z=1-x-y$



$y \to 0$ to $1-x$

$x \to 0$ to 1



$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} \, dz \, dy \, dx$$

$$\left(\int \frac{1}{z^3} = \frac{z^{-2}}{-2} \right)$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(1+x+y+z)^{-2}}{-2} \right]_0^{1-x-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{2^{-2}}{-2} - \frac{(1+x+y)^{-2}}{-2} \right) dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{(1+x+y)^2} - \frac{1}{4} \right) dy \, dx$$

$$\int \frac{1}{y^2} = \frac{y^{-1}}{-1}$$

$$= \frac{1}{2} \int_0^1 \left[\frac{(1+x+y)^{-1}}{-1} - \frac{y}{4} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{2^{-1}}{-1} - \frac{1-x}{4} - \frac{(1+x)^{-1}}{-1} \right) dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{1+x} - \frac{1-x}{4} - \frac{1}{2}\right) dx$$

$$= \frac{1}{2}\left[\log(1+x) - \frac{1}{4}x + \frac{x^2}{8} - \frac{1}{2}x\right]_0^1$$

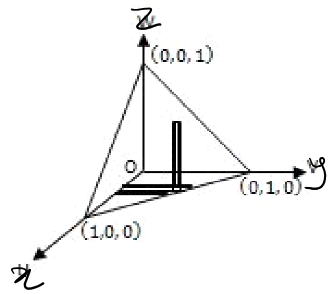
$$I = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

4. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$ (HW)

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dz\,dy\,dx = \frac{1}{8} \text{ (Ans)}$$

5. Evaluate in terms of Gamma function $\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$ throughout the volume of the tetrahedron $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1.$

$$I = \int_0^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\,dz\,dy\,dx$$



$$= \int_0^1 \int_{y=0}^{1-x} x^{l-1} y^{m-1} \left(\frac{z^n}{n}\right)_0^{1-x-y} dy\,dx$$

$$= \frac{1}{n} \int^{1-x} x^{l-1} y^{m-1} (1-x-y)^n\,dy\,dx$$

$$= \frac{1}{n}\int_0 \int_0 x^{l-1} y^{m-1} (1-x-y)^n \, dy \, dx$$

put $1-x = a$ (Note this)

$$= \frac{1}{n}\int_0^1 x^{l-1} \left[\int_0^a y^{m-1} (a-y)^n \, dy \right] dx$$

put $y = at$ $\quad dy = a\,dt$

when $y = 0, \quad t = 0$

$y = a \quad t = 1$

$$= \frac{1}{n}\int_0^1 x^{l-1} \left[\int_0^1 (at)^{m-1} (a-at)^n \, a\,dt \right] dx$$

$$= \frac{1}{n}\int_0^1 x^{l-1} a^{m+n} \left[\int_0^1 t^{m-1} (1-t)^n \, dt \right] dx$$

$$= \frac{1}{n}\int_0^1 x^{l-1} a^{m+n} \, B(m, n+1) \, dx$$

$$= \frac{B(m,n+1)}{n}\int_0^1 x^{l-1} (1-x)^{m+n} \, dx$$

$$= \frac{B(m,n+1)}{n} \cdot B(l, m+n+1)$$

$$= \frac{1}{n} \cdot \frac{\Gamma m \, \Gamma(n+1)}{\Gamma(m+n+1)} \cdot \frac{\Gamma l \, \Gamma(m+n+1)}{\Gamma(m+n+l+1)} = \frac{1}{n} \cdot \frac{\Gamma m \, \Gamma l \, \Gamma(n+1)}{\Gamma(l+m+n+1)} \checkmark$$

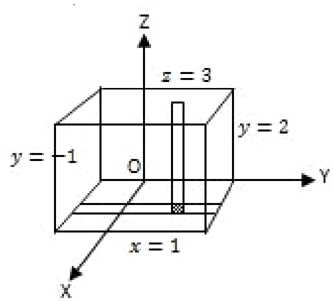
$$n \quad \overline{\Gamma(m+n+1)} \quad \frac{\quad}{\Gamma(m+n+l+1)} = \frac{1}{n} \cdot \frac{\cdots \Gamma(l)\,\Gamma(n+1)}{\Gamma(m+n+l+1)}$$

$$= \frac{1}{n} \cdot \frac{\Gamma(m)\,\Gamma(l) \cdot n\,\Gamma(n)}{(l+m+n)\,\Gamma(l+m+n)} = \frac{1}{(l+m+n)} \cdot \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}$$

6. Evaluate the integral $\iiint_v xyz^2\, dv$ over the region bounded by the planes $x = 0, x = 1, y = -1, y = 2,$ $z = 0, z = 3$

$$I = \int_{x=0}^{1} \int_{y=-1}^{2} \int_{z=0}^{3} xyz^2 \, dx\, dy\, dz$$

$$= \left(\int_0^1 x\, dx \right) \left(\int_{-1}^{2} y\, dy \right) \left(\int_0^3 z^2\, dz \right)$$



$$= \left(\frac{x^2}{2} \right)_0^1 \left(\frac{y^2}{2} \right)_{-1}^{2} \left(\frac{z^3}{3} \right)_0^3$$

$$= \left(\frac{1}{2} \right) \left(2 - \frac{1}{2} \right) (9) = \frac{1}{2} \left(\frac{3}{2} \right) (9) = \frac{27}{4}$$