MASS OF LAMINA

Tuesday, June 1, 2021 11:30 AM

lamina → plate

density = $\frac{\text{mass}}{\text{Area}} \Rightarrow$ mass = density × Area

ρ : (rho)

(a) For a plane lamina of area A, if the density at a point $P(x,y)$ be $\rho = f(x,y)$, then its total mass M is given by

$$M = \iint_A \rho \, dx\, dy = \iint_A f(x,y) dx\, dy$$

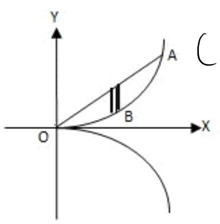
$A = \iint dx\,dy$

(b) In polar coordinates, if the density at a point $P(r,\theta)$ be $\rho = f(r,\theta)$ then its total mass M is given by

$$M = \iint_A \rho \, r\, dr\, d\theta = \iint_A f(r,\theta) r \, dr\, d\theta$$

1. Find the mass of the lamina bounded by the curve $ay^2 = x^3$ and the line $by = x$ if the density at a point varies as the distance of the point from the x-axis.

Solution: The curves intersect at $A\left(\frac{a}{b^2}, \frac{a}{b^3}\right)$. The lamina is the area OBA.



$A\left(\frac{a}{b^2}, \frac{a}{b^3}\right)$

Point of intersection

$ay^2 = x^3$

$by = x$

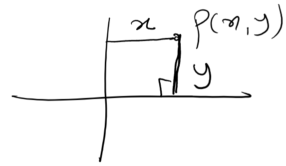
$ay^2 = b^3y^3$

$y = \frac{a}{b^3} \Rightarrow x = \frac{a}{b^2}$

$y \to ay^2 = x^3$ to $by = x$

$y = \left(\frac{x^3}{a}\right)^{1/2}$ to $y = \frac{x}{b}$

$x \to 0$ to $\frac{a}{b^2}$



Let $P(x,y)$ be any point on the lamina

$\rho \propto y \Rightarrow \rho = ky$

On the curve $OBA, y = x^{3/2}/\sqrt{a}$ and on the line $OA, y = x/b$.
The surface density is given by $\rho = ky$. Taking the elementary strip parallel to the y −axis, mass of the lamina

$$= k \int_0^{a/b^2} \int_{x^{3/2}/\sqrt{a}}^{x/b} y \, dx dy$$

$$= k \int_0^{a/b^2} \left[\frac{y^2}{2}\right]_{\frac{x^{3/2}}{\sqrt{a}}}^{\frac{x}{b}} dx = \frac{k}{2}\int_0^{a/b^2} \left(\frac{x^2}{b^2} - \frac{x^3}{a}\right) dx$$

$$= \frac{k}{2}\left[\frac{x^3}{3b^2} - \frac{x^4}{4a}\right]_0^{a/b^2} = \frac{k}{2}\left[\frac{a^3}{3b^8} - \frac{a^4}{4ab^8}\right]$$

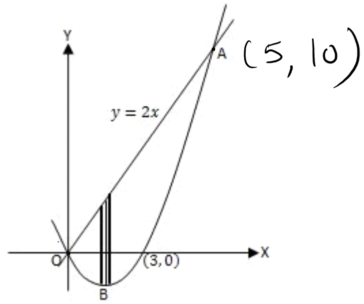
$$\overline{2}\left[3b^2 \quad 4a\right]_0 \qquad \left[3b \quad 4ab\right]$$

$$M = \frac{k}{24} \cdot \frac{a^3}{b^8}$$

2. A lamina is bounded by $y = x^2 - 3x$ and $y = 2x$. If the density at any point is given by $(24/25)xy$. Find the mass of the lamina

Solution: The curve $y = x^2 - 3x$ i.e. $y + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$ is a parabola intersecting the x −axis in $x = 0$ and $x = 3$

The line $y = 2x$ intesects this parabola at $x^2 - 3x = 2x$ i.e. $x^2 - 5x = 0$ i.e. at $x = 0, x = 5$.



$y \to$ Parabola to line

$y \to x^2 - 3x$ to $2x$

$x \to 0$ to A

0 to 5

Therefore, points of intersection are $(0,0)$ and $(5,10)$. The lamina is the area OAB. Taking the elementary strip parallel to the y −axis, mass of lamina

$$= \int_0^5 \int_{x^2-3x}^{2x} \left(\frac{24}{25}\right) xy \, dx dy$$

$$= \frac{24}{25} \int_0^5 x \left(\frac{y^2}{2}\right)_{x^2-3x}^{2x} dx = \frac{12}{25} \int_0^5 x\left(4x^2 - (x^2-3x)^2\right) dx$$

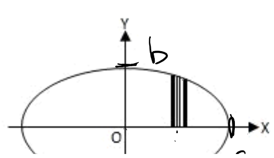
$$= \frac{12}{25} \int_0^5 x\left(4x^2 - x^4 + 6x^3 - 9x^2\right) dx$$

$$= \frac{12}{25} \int_0^5 \left(6x^4 - x^5 - 5x^3\right) dx$$

$$= \frac{12}{25}\left[\frac{6x^5}{5} - \frac{x^6}{6} - \frac{5x^4}{4}\right]_0^5 = 175.$$

3. Find the mass of the lamina in the form of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if the density at any point varies as the product of the distances from the axes of the ellipse.

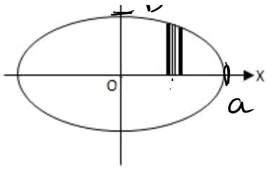
Solution:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$\rho \propto$ product of distances from axes of ellipse



$y = \frac{b}{a}\sqrt{a^2 - x^2}$

axes of ellipse

$\rho \propto$ product of distance from x axis and y axis

$\rho \propto xy \Rightarrow \rho = kxy$

Mass of the lamina $= 4\iint \rho \, dxdy$

$$= 4k \iint xy \, dxdy$$

$$= 4k \int_0^a \int_0^{b\sqrt{a^2-x^2}/a} xy \, dxdy$$

$$= 4k \int_0^a x \cdot \left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

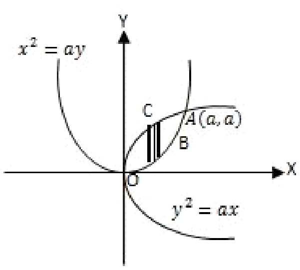
$$= \frac{4k}{2} \int_0^a x \cdot \left(\frac{b^2}{a^2}(a^2 - x^2)\right) dx$$

$$= 2k \cdot \frac{b^2}{a^2} \int_0^a (a^2 x - x^3) \, dx$$

$$= \frac{2kb^2}{a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{ka^2b^2}{2}$$

4. Find the mass of the lamina bounded by the curves $y^2 = ax$ and $x^2 = ay$ if the density of the lamina at any point varies as the square of its distance from the origin.

Solution:



if $P(x, y)$ is any point on the lamina

then distance from origin $= \sqrt{x^2 + y^2}$

$\therefore \rho \propto (x^2 + y^2)$

$\therefore \rho = k(x^2 + y^2)$

$y \to \frac{x^2}{a}$ to $\sqrt{ax}$

$x \to 0$ to a

The two curves intersect at $A(a, a)$. The lamina is the area $OBACO$.

On the curve OCA, $y = \sqrt{ax}$ and on the curve OBA, $y = x^2/a$. The surface density is give by $\rho = k(x^2 + y^2)$.

Taking the elementary strip parallel to the y –axis, the mass of the lamina

$$= k \int_0^a \int_{x^2/a}^{\sqrt{ax}} (x^2 + y^2)\, dxdy$$

$$= k \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_{\frac{x^2}{a}}^{\sqrt{ax}} dx$$

$$= k \int_0^a \left[x^2 \cdot \sqrt{ax} + \frac{(ax)\sqrt{ax}}{3} - x^2 \cdot \frac{x^2}{a} - \frac{1}{3} \cdot \frac{x^6}{a^3} \right] dx$$

$$= k \int_0^a \left(\sqrt{a}\, x^{5/2} + \frac{a\sqrt{a}}{3} x^{3/2} - \frac{1}{a} x^4 - \frac{1}{3a^3} x^6 \right) dx$$

$$= k \left[\sqrt{a} \cdot \frac{x^{7/2}}{7/2} + \frac{a\sqrt{a}}{3} \cdot \frac{x^{5/2}}{5/2} - \frac{1}{a} \frac{x^5}{5} - \frac{1}{3a^3} \cdot \frac{x^7}{7} \right]_0^a$$

$$= k \left[\frac{2}{7} a^4 + \frac{2}{15} a^4 - \frac{a^4}{5} - \frac{a^4}{21} \right] = \frac{6ka^4}{35}$$

5. The density of a uniform circular lamina of radius a varies as the square of its distance from a fixed point on the circumference of the circle. Find the mass of the lamina.

Solution: Let the fixed point on the circumferences of the circle be the origin and the diameter through it be the x −axis. Then the polar equation of the circle is $r = 2a\cos\theta$.

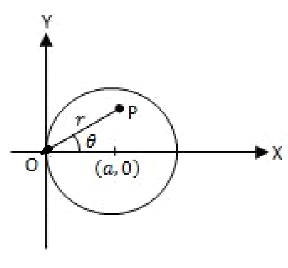
$(x-a)^2 + y^2 = a^2$

$P(x,y)$ is any point on lamina

$\rho \propto$ sq. distance of P from a fixed point on the circumference

$\rho \propto$ sq. of distance of P from origin

$\rho = kr^2$



The density at any point $P(r,\theta)$ is $= kr^2$.

Hence, Mass of lamina $= 2\int_0^{\pi/2}\int_0^{2a\cos\theta}(kr^2)r\,drd\theta$

$$= 2\int_0^{\pi/2} k \left(\frac{r^4}{4} \right)_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{2} k \int^{\pi/2} 16a^4 \cos^4\theta \, d\theta$$

$$= \frac{1}{2} k \int_0 16 a^4 \cos^4\theta \, d\theta$$

$$= 8 k a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 8 k a^4 \cdot \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{3}{2} k a^4 \pi$$

6. The density at any point of a cardioide $r = a(1 + cos\theta)$ varies as the square of its distance from its axis of symmetry. Find its mass.

Solution: Let $P(r, \theta)$ be any point on the given cardioide.

The distance of P from the axis is $r \sin\theta$.

The density at any point $P(r, \theta)$ is $\rho = kr^2 \sin^2\theta$

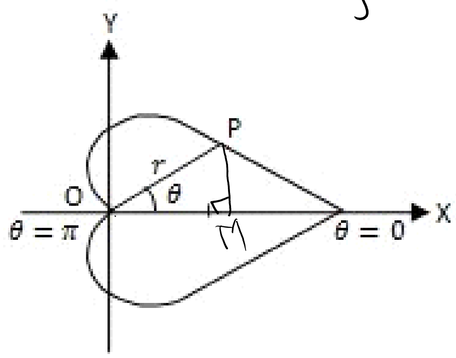
$P(r, \theta)$ is any point

$\rho \propto$ sq. distance of P from x-axis

$\rho \propto PM^2$

$\rho \propto (r \sin\theta)^2$

$\rho = k r^2 \sin^2\theta$



Mass of the lamina $= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (kr^2 \sin^2\theta) r \, dr d\theta$

$$= 2\int_0^{\pi} k \sin^2\theta \left(\frac{r^4}{4}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{k}{2}\int_0^{\pi} \sin^2\theta \left(a^4 (1+\cos\theta)^4\right) d\theta$$

$$= \frac{ka^4}{2}\int_0^{\pi} \left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^2 \left[2\cos^2\frac{\theta}{2}\right]^4 d\theta$$

$$= ka^4 \cdot 2^6 \int_0^{\pi} \sin^2\frac{\theta}{2} \cos^{10}\theta \, d\theta$$

$$= \frac{ka^4}{2} \cdot 2^6 \int_0 \sin^2\frac{\theta}{2} \cos^{10}\frac{\theta}{2}\, d\theta$$

put $\frac{\theta}{2} = t$ $\quad d\theta = 2dt$

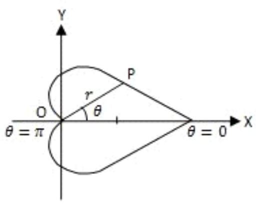
θ	0	π
t	0	$\pi/2$

$$M = ka^4 \cdot 32 \cdot \int_0^{\pi/2} \sin^2 t \cos^{10} t \cdot 2\, dt$$

$$= ka^4 \cdot 32 \cdot 2 \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{11}{2}\right) = \frac{21}{32} ka^4 \pi$$

7. Find the mass of the lamina in the form of a cardioide $r = a(1 + cos\theta)$ if the density of mass at a point varies as the distance from the pole.

Solution:



$\rho \propto$ distance from the pole (origin)

$P(r, \theta)$ is any point

$\rho \propto r$

$\rho = kr$

Mass $= 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} (kr) r\, dr d\theta$

$$= 2\int_0^{\pi} k\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2k}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 d\theta$$

$$= \frac{2ka^3}{3}\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= \frac{16ka^3}{3}\int_0^{\pi} \cos^6\frac{\theta}{2}\, d\theta$$

put θ ... $d\theta = 2dt$

θ	0	π

put $\frac{\theta}{2} = t \qquad d\theta = 2\,dt$

θ	0	π
t	0	$\pi/2$

$$M = \frac{16ka^3}{3} \int_0^{\pi/2} \cos^6 t \,(2\,dt)$$

$$= \frac{16ka^3}{3} \cdot B\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{5}{3} k a^3 \pi$$

8. Find the mass of a plate in the form of one loop of lemniscate $r^2 = a^2 \cos 2\theta$ if the density varies as the square of the distance from the pole.

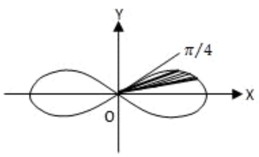
Solution:

$P(r,\theta)$ is any point

$\rho \propto$ sq. of distance of P from pole (origin)

$\rho \propto r^2$

$\rho = k r^2$



Mass $= 2\int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} kr^2 \cdot r\, dr d\theta$

$$M = 2\int_0^{\pi/4} k \left(\frac{r^4}{4}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{2k}{4} \int_0^{\pi/4} a^4 \cos^2 2\theta \, d\theta$$

put $2\theta = t \qquad d\theta = \frac{dt}{2}$

θ	0	$\pi/4$
t	0	$\pi/2$

$$M = \frac{2ka^4}{4} \int_0^{\pi/2} \cos^2 t \, \frac{dt}{2}$$

$$= \frac{ka^4}{4} \int_0^{\pi/2} \cos^2 t \, dt$$

$$= \frac{ka^4}{4} \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{ka^4}{8} \cdot \frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2}$$

$$= \frac{ka^4}{8} \cdot \frac{\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 2} = \frac{ka^4\pi}{16}$$

9. Find the mass of the lamina bounded by the curves $ay^2 = x^3$ and $by = x$, if the density at a point varies as

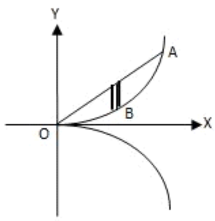
(i) the cube of distance of the point from the x −axis,

(ii) the square of the distance of the point from the x −axis

Solution:

(i) $\rho \propto y^3 \Rightarrow \rho = ky^3$

(ii) $\rho \propto y^2 \Rightarrow \rho = ky^2$



(i) $\rho = ky^3$,

$\therefore M \;= k\int_0^{a/b^2}\int_{x^{3/2}/\sqrt{a}}^{x/b} y^3\,dydx$

$$= k\int_0^{a/b^2}\left(\frac{y^4}{4}\right)_{\frac{x^{3/2}}{\sqrt{a}}}^{x/b} dx = \frac{k}{4}\int_0^{a/b^2}\left(\frac{x^4}{b^4} - \frac{x^6}{a^2}\right)dx$$

$$= \frac{k}{4}\left[\frac{x^5}{5b^4} - \frac{x^7}{7a^2}\right]_0^{\frac{a}{b^2}} = \frac{k}{4}\left[\frac{a^5}{5b^{14}} - \frac{a^7}{7a^2b^{14}}\right]$$

$$= \frac{k}{4}\cdot\frac{a^5}{b^{14}}\left[\frac{1}{5} - \frac{1}{7}\right] = \frac{k}{70}\,\frac{a^5}{b^{14}}$$

(ii) $\rho = ky^2$,

$\therefore M \;= k\int_0^{a/b^2}\int_{x^{3/2}/\sqrt{a}}^{x/b} y^2\,dydx$

$$= k\int_0^{a/b^2}\left(\frac{y^3}{3}\right)_{x^{3/2}/\sqrt{a}}^{x/b} dx$$

a/b^2

$$= \frac{K}{3}\int_0^{a/b^2}\left(\frac{x^3}{b^3} - \frac{x^{9/2}}{a\sqrt{a}}\right)dx$$

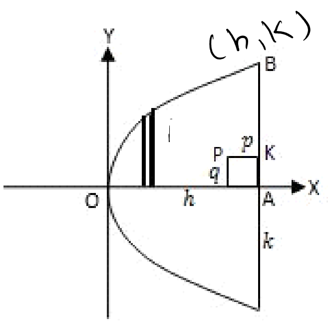
$$= \frac{K}{3}\left[\frac{x^4}{4b^3} - \frac{x^{11/2}}{a\sqrt{a}\cdot\frac{11}{2}}\right]_0^{a/b^2}$$

$$= \frac{K}{3}\left[\frac{a^4}{4b^{11}} - \frac{2a^{11/2}}{11a\sqrt{a}\,b^{11}}\right] = \frac{K}{3}\cdot\frac{a^4}{b^{11}}\left[\frac{1}{4} - \frac{2}{11}\right]$$

$$M = \frac{K}{44}\cdot\frac{a^4}{b^{11}}$$

10. A lamina in the form of a parabolic segment of mass M, height h and base $2k$ has density at a point given by λpq^3 per unit area where p, q are distances from the base and axis respeactively. Find the value of λ.

Solution:



B(h, k) lies on parabola

∴ it satisfies eqn of parabola

$k^2 = 4ah$

$\rightarrow y^2 = \frac{k^2}{h}x$

Let the parabolic segment be as shown in the figure. Let the equation of the parabola be $y^2 = 4ax$.
Since the point $B(h,k)$ lies on the parabola $\quad k^2 = 4ah; 4a = k^2/h$;
the equation is $y^2 = \frac{k^2}{h}x$

∴ If $P(x,y)$ is any point on the lamina, then the distances p, q are as shown in the figure.

∴ $x + p = h$ i.e. $p = h - x; q = y$

Mass of the lamina $M = 2\int_0^h\int_0^{k\sqrt{x/h}} \lambda pq^3\,dxdy$

$$= 2\lambda\int_0^h\int_0^{k\sqrt{x/h}}(h-x)\,y^3\,dy\,dx$$

$$= 2\lambda \int_0^h (h-x)\left(\frac{y^4}{4}\right)_0^{k\sqrt{\frac{x}{h}}} dx$$

$$= \frac{2\lambda}{4}\int_0^h (h-x)\, k^4 \frac{x^2}{h^2}\, dx$$

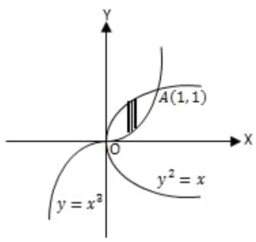
$$= \frac{\lambda}{2} \cdot \frac{k^4}{h^2}\int_0^h (hx^2 - x^3)\, dx$$

$$M = \frac{\lambda k^4}{2h^2}\left[\frac{hx^3}{3} - \frac{x^4}{4}\right]_0^h = \frac{\lambda k^4}{2h^2}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$M = \frac{\lambda k^4 h^2}{24} \Rightarrow \lambda = \frac{24M}{k^4 h^2}$$

11. Find ~~the~~ by double integration the mass of a thin plate bounded by $y^2 = x$ and $y = x^3$ if the density at any point varies as the square of its distance from the origin

Solution:



$y \to x^3$ to $\sqrt{x}$

$x \to 0$ to 1

pt of intersection

$(0,0)$ & $(1,1)$

$\rho \propto (x^2 + y^2)$

$\rho = k(x^2+y^2)$

Clearly the curves intersect at $A(1,1)$

Mass $= \iint \rho\, dxdy$

$$= \int_0^1 \int_{x^3}^{\sqrt{x}} k(x^2+y^2)\, dxdy$$

$$= k\int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_{x^3}^{\sqrt{x}} dx$$

$$= K\int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^5 - \frac{x^9}{3}\right) dx$$

$$= K\left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{15/2} - \frac{x^6}{6} - \frac{x^{10}}{30}\right]_0^1$$

$$= K\left[\frac{2}{7} + \frac{2}{15} - \frac{1}{6} - \frac{1}{30}\right] = \frac{23}{105}K.$$