AREA

Monday, May 31, 2021 11:30 AM

(a) The area enclosed by two plane curves $y = f_1(x)$ and $y = f_2(x)$ intersecting in $A(a, c)$ and $B\ (b, d)$ is $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dx\, dy$

(b) The area enclosed by two plane curves $r = f_1(\theta)$ and $r = f_2(\theta)$ intersecting in $A(r_1, \alpha)$ and $B(r_2, \beta)$ is $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\, d\theta\, dr$

SOME SOLVED EXAMPLES:

1. Find by double integration the area enclosed $y^2 = x^3$ and $y = x$

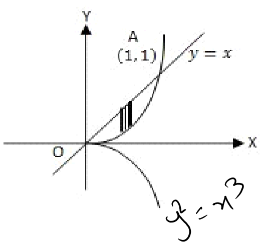
Solution: The two curves intersect at the origin $O(0,0)$ and $A(1,1)$

$$\left.\begin{aligned} y^2 &= x^3 \\ y &= x \end{aligned}\right\} \Rightarrow x^2 = x^3$$

$$\Rightarrow x^2(x-1) = 0$$

$x = 0$ or $x = 1$

$y = 0$ or $y = 1$



Consider a strip parallel to the y —axis. on this strip y varies from $x^{3/2}$ to x.
And the strip moves from 0 to 1

$\therefore A \ = \int_0^1 \int_{x^{3/2}}^{x} dydx$

$$= \int_0^1 (y)_{x^{3/2}}^{x} dx = \int_0^1 \left(x - x^{3/2}\right) dx = \left(\frac{x^2}{2} - \frac{x^{5/2}}{5/2}\right)_0^1 = \frac{1}{10}$$

2. Find the area between parabola $y = x^2 - 6x + 3$ and the line $y = 2x - 9$

Solution: We have $y = (x-3)^2 - 6$ i.e. $y + 6 = (x-3)^2$.

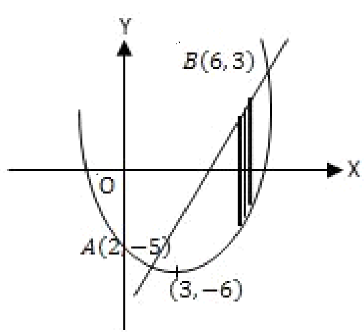
It is a parabola with vertex at $(3, -6)$ and opening upwards.

The line intersects the parabola where $x^2 - 6x + 3 = 2x - 9$ i.e. $x^2 - 8x + 12 = 0$

i.e. $(x-6)(x-2) = 0$ i.e. when $x = 6, x = 2$.

When $x = 6,\ y = 12 - 9 = 3$; when $x = 2, y = 4 - 9 = -5$.

The points of intersection are $B(6,3), A\ (2, -5)$



To find the area consider a strip parallel to the y —axis.

On this strip y varies from $y = x^2 - 6x + 3$ to $y = 2x - 9$.

Then x varies from $x = 2$ to $x = 6$

$\therefore A \ = \int_{x=2}^{6} \int_{y=x^2-6x+3}^{2x-9} dydx$

$$= \int^6 (y)_{x^2-6x+3}^{2x-9} dx = \int^6 (2x - 9 - x^2 + 6x - 3)\, dx$$

$$= \int_2^6 (y)_{x^2-6x+3}^{2x-9} dx = \int_2^6 (2x-9-x^2+6x-3)\,dx$$

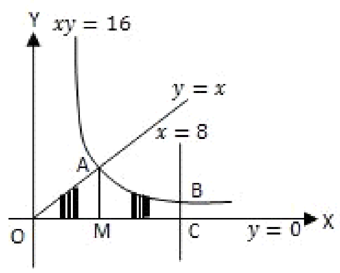
$$= \int_2^6 (-x^2+8x-12)\,dx = \left(\frac{-x^3}{3} + \frac{8x^2}{2} - 12x\right)_2^6$$

$$= \frac{-6^3}{3} + 4(6)^2 - (12\times 6) + \frac{2^3}{3} - 4(2)^2 + (12\times 2)$$

$$A = \frac{32}{3}$$

3. Sketch the region bounded by the curves $xy = 16, y = x, x = 8$ and $y = 0$.
Express the area of this region as a double integral in two ways

Solution: The curve $xy = 16$ is a ractangular hyperbola.
$y = x$ is a line passing through the origin and equally inclined to the axes.
$y = 0$ is the x —axis and $x = 8$ is a line parallel to the y —axis.



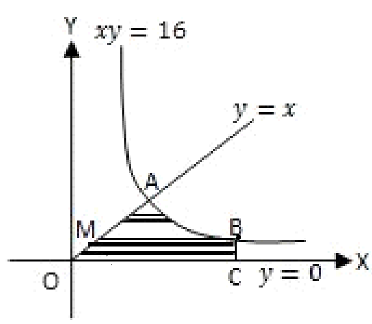
point of intersection of $xy = 16$
and $y = x$
$\Rightarrow x^2 = 16 \Rightarrow x = 4$
$y = 4.$
$A(4,4)$

Thus, the region is $OABC$
The vertices of the figure are $O(0,0), C(8,0), B(8,2), A(4,4)$.
If we drop the perpendicular AM, then M is $(4,0)$
If we take a strip parallel to the y —axis, then the area divided into
two regions OMA and $AMCB$
$\therefore$ Area $= \int_0^4 \int_{y=0}^{x} dxdy + \int_4^8 \int_{y=0}^{16/x} dxdy$

point of intersection of $xy = 16$ and $x = 8$
$y = 2$
$B \equiv (8,2)$

$AMB \rightarrow y = x$ to hyperbola

$OMBC \rightarrow y = x$ to $x = 8$



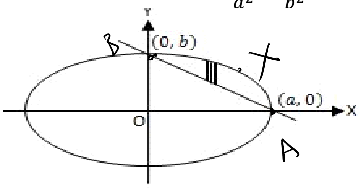
If we take a strip parallel to the x −axis, then the area is divided into two regions $OMBC$ and MBA where M is the point of intesection of a line parallel to the x −axis through B

∴ Area $= \int_0^2 \int_{x=y}^{8} dxdy + \int_2^4 \int_{x=y}^{y/16} dxdy$

(Evaluation can be asked).

4. Find by double integration the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$

Solution: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$ are shown in the figure



$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{y}{b} = 1 - \frac{x}{a} \Rightarrow y = b\left(1 - \frac{x}{a}\right)$

$= \frac{b}{a}(a-x)$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \frac{b}{a}\left(\sqrt{a^2-x^2}\right)$

$\therefore A \quad = \int_0^a \int_{(b/a)(a-x)}^{(b/a)\sqrt{a^2-x^2}} dydx$

$= \int_0^a [y]_{\frac{b}{a}(a-x)}^{\frac{b}{a}(\sqrt{a^2-x^2})} dx$

$= \int_0^a \frac{b}{a}\left[\sqrt{a^2-x^2} - (a-x)\right] dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - ax + \frac{x^2}{2}\right]_0^a = \frac{ba}{4}(\pi - 2)$

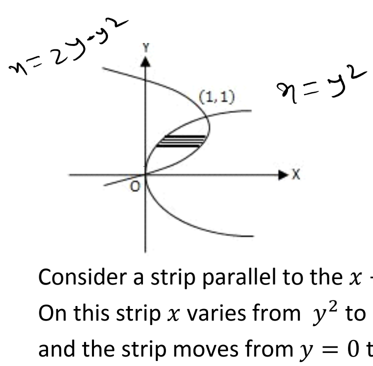
5. Using double integration find the area bounded by the parabolas $x = y^2, x = 2y - y^2$

Solution: The parabola $y^2 = x$ has vertex at the origin.

The parabola $y^2 - 2y = -x$ i.e. $(y-1)^2 = -(x-1)$ has vertex at $(1, 1)$.

The two parabolas intersect where $y^2 = 2y - y^2$ i.e. $2y(y-1) = 0$ ∴ $y = 0, y = 1$

The points of intersection are $(0, 0), (1, 1)$



Consider a strip parallel to the x −axis.

On this strip x varies from y^2 to $2y - y^2$

and the strip moves from $y = 0$ to $y = 1$

Hence, $A \quad = \int_0^1 \int_{y^2}^{2y-y^2} dxdy$

$= \int^1 (y)_{y^2}^{2y-y^2} dy = \int^1 (2y - 2y^2) dy = \left(y^2 - \frac{2y^3}{3}\right)^1 = \frac{1}{3}$

$$= \int_0^1 (y)_{y^2}^{2y-y^2} dy = \int_0^1 (2y - 2y^2)\, dy = \left(y^2 - \frac{2y^3}{3}\right)_0^1 = \frac{1}{3}$$

6. Find by double integration the area included between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$

Solution: We have $y = 3x^2 - x - 3$

i.e. $y + 3 + \frac{1}{12} = 3\left(x^2 - \frac{1}{3}x + \frac{1}{36}\right)$

i.e. $y + \frac{37}{12} = 3\left(x - \frac{1}{6}\right)^2$

which is a parabola with vertex at $(1/6, -37/12)$ and opening upwards

and $y = -2x^2 + 4x + 7$

i.e. $y - 7 = -2(x^2 - 2x)$

i.e. $y - 9 = -2(x-1)^2$

which is a parabola with vertex at $(1, 9)$ and opening downwards

The two curves intersect when $3x^2 - x - 3 = -2x^2 + 4x + 7$

$\therefore 5x^2 - 5x - 10 = 0$

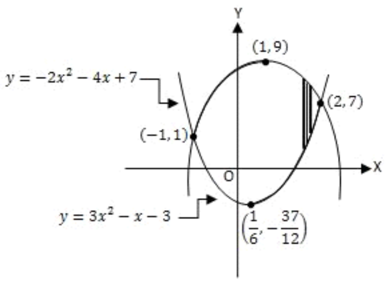
$\therefore x^2 - x - 2 = 0$

$\therefore (x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

When $x = -1, y = +1$ and when $x = 2, y = 7$.

Thus, the two curves intersect in $(-1, 1)$ and $(2, 7)$



Now, consider a strip parallel to the y −axis.

On this strip y varies from $3x^2 - x - 3$ to $-2x^2 + 4x + 7$.

Then x −varies from $x = -1$ to $x = 2$

$\therefore A = \int_{-1}^{2} \int_{3x^2-x-3}^{-2x^2+4x+7} dy dx$

$$= \int_{-1}^{2} \left[y\right]_{3x^2-x-3}^{-2x^2+4x+7} dx$$

$$= \int_{-1}^{2} \left(-2x^2 + 4x + 7 - 3x^2 + x + 3\right) dx$$

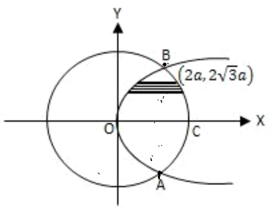
$$= \int_{-1}^{2} \left(-5x^2 + 5x + 10\right) dx$$

$$= \left(-\frac{5x^3}{3} + \frac{5x^2}{2} + 10x\right)_{-1}^{2} = \left(-\frac{40}{3} + 10 + 20\right) - \left(\frac{5}{3} + \frac{5}{2} - 10\right)$$

$$= -15 + 40 - \frac{5}{2} = \frac{45}{2}$$

7. Find the larger of the two areas into which the circle $x^2 + y^2 = 16a^2$ is divided by the parabola $y^2 = 6ax$

Solution: We shall first find the common area $AOBCA$. centre $(0,0)$ radius $= 4a$.



$x \to$ parabola ~~to~~ circle

The points of intersection are given by

$x^2 + 6ax - 16a^2 = 0 \quad \therefore (x + 8a)(x - 2a) = 0$

$\therefore x = 2a$

$\therefore y^2 = 12a^2 \quad \therefore y = 2\sqrt{3} \cdot a$

Hence, B is $\left(2a, 2\sqrt{3} \cdot a\right)$

$\therefore$ Area $= 2\int_0^{2\sqrt{3}\cdot a} \int_{x=y^2/6a}^{\sqrt{16a^2-y^2}} dxdy$

$$= 2\int_0^{2\sqrt{3}a} [x]_{\frac{y^2}{6a}}^{\sqrt{16a^2-y^2}} dy$$

$$= 2\int_0^{2\sqrt{3}a} \left(\sqrt{16a^2-y^2} - \frac{y^2}{6a}\right) dy$$

$$= 2\left[\frac{y}{2}\sqrt{16a^2-y^2} + \frac{16a^2}{2}\sin^{-1}\left(\frac{y}{4a}\right) - \frac{y^3}{18a}\right]_0^{2\sqrt{3}a}$$

$$= \left[2\sqrt{3}a \cdot 2a + 16a^2 \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{24\sqrt{3}}{9a} \cdot a^3\right]$$

$$= 4\sqrt{3}a^2 + 16a^2 \frac{\pi}{3} - \frac{8}{3}\sqrt{3}a^2$$

$$= \frac{4}{3}(4\pi + \sqrt{3})a^2$$

But area of the circle $= \pi 16a^2$

$\therefore$ Required area $= \pi 16a^2 - \frac{4}{3}\left(4\pi + \sqrt{3}\right)a^2 = \frac{4}{3}\left(8\pi - \sqrt{3}\right)a^2$

8. Find by double integration the area common to the circles $x^2 + y^2 - 4y = 0$ and $x^2 + y^2 - 4x - 4y + 4 = 0$

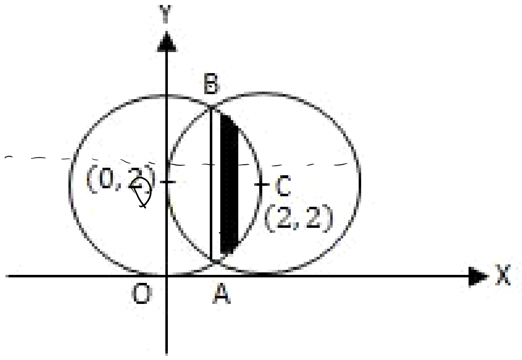
Solution: The equation $x^2 + y^2 - 4y = 0$ can be written as $x^2 + (y-2)^2 = 2^2$.

Its Centre is $(0, 2)$ and radius $= 2$.

And the equation $x^2 + y^2 - 4x - 4y + 4 = 0$ can be written as $(x-2)^2 + (y-2)^2 = 2^2$.

Its Centre is $(2, 2)$ and radius $= 2$

By subtraction, we see that the circles intersect at points where $x = 1$



$x^2 + y^2 - 4y = 0$

$y = \frac{4 \pm \sqrt{16 - 4x^2}}{2}$

$= 2 \pm \sqrt{4 - x^2}$

lower $y \to 2 - \sqrt{4 - x^2}$

upper $y \to 2 + \sqrt{4 - x^2}$

Consider a strip parallel to the y –axis.

Then on the circle on the left i.e. on $x^2 + y^2 - 4y = 0$ i.e. on $y = \frac{4 \pm \sqrt{16 - 4x^2}}{2}$,

y varies from $2 - \sqrt{4 - x^2}$ to $2 + \sqrt{4 - x^2}$

$\therefore$ Required area $= 2$ (area ABC) by symmetry

$= 2\int_1^2 \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dy\,dx$

$$= 2\int_1^2 \left(y\right)_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} dx$$

$$= 2\int_1^2 2\sqrt{4-x^2}\, dx$$

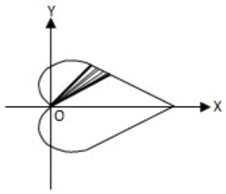
$$= 4\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= 4 \cdot \left[2 \cdot \frac{\pi}{2} - \left(\frac{\sqrt{3}}{2} + 2 \cdot \frac{\pi}{6}\right)\right]$$

$$= 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

9. Find the area of the cardioide $r = a(1 + \cos\theta)$

Solution:



For the cardioid r varies from 0 to $a(1+\cos\theta)$ and θ varies from 0 to π above the x –axis

$\therefore$ Area= $2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,drd\theta$

$$= 2\int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta = \int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$= a^2\int_0^{\pi} 4\cos^4\left(\frac{\theta}{2}\right)d\theta$$

put $\frac{\theta}{2} = t$ $\quad d\theta = 2dt$

θ	0	π
t	0	$\pi/2$

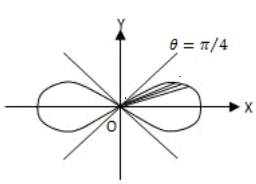
$$= 4a^2\int_0^{\pi/2} \cos^4 t\,(2dt) = 8a^2\int_0^{\pi/2}\cos^4 t\,dt$$

$$= 8a^2\cdot\frac{1}{2}B\left(\frac{5}{2},\frac{1}{2}\right) = \frac{3}{2}\pi a^2$$

10. Find the total area enclosed by the lemniscate of Bernoulli $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$

Solution: We transform the equation to polar form by putting $x = r\cos\theta$, $y = r\sin\theta$

$\therefore r^4 = a^2r^2\cos 2\theta$ i.e. $r^2 = a^2\cos 2\theta$



$r \to 0$ to $a\sqrt{\cos 2\theta}$

$\theta \to 0$ to $\pi/4$

Now, consider a small radial strip in the upper half of one loop

$\therefore A \;= 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,drd\theta$

$$= 4\int_0^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} a^2\cos 2\theta \, d\theta = 2a^2\int_0^{\pi/4} \cos 2\theta \, d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2$$

11. Find the area inside the circle $r = a\,sin\theta$ and outside the cardioide $r = a\,(1 - cos\theta\,)$.

Solution: The circle and the cardioide intersect where $a\sin\theta = a(1-\cos\theta)$

i.e. $2\sin(\theta/2)\cos(\theta/2) = 2\sin^2(\theta/2)$

i.e. $\sin\theta/2\,[\sin(\theta/2) - \cos(\theta/2)] = 0$

When $\sin\theta/2 = 0 \;\therefore \theta = 0$

When $\sin\frac{\theta}{2} - \cos\frac{\theta}{2} = 0, \therefore \frac{\theta}{2} = \frac{\pi}{4} \;\therefore \theta = \frac{\pi}{2}$

$r = a\sin\theta$

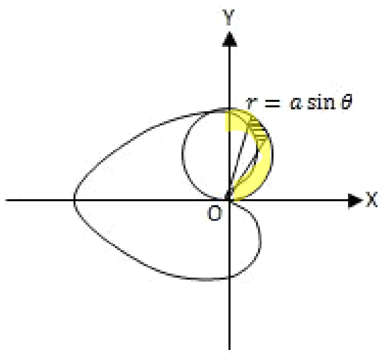
$r^2 = ar\sin\theta$

$x^2+y^2 = ay$

$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$

centre $\left(0, \frac{a}{2}\right)$

radius $\rightarrow \frac{a}{2}$



$r \rightarrow$ varies from cardioide to circle

Now, consider a radial strip in the region of integration.

On this strip r varies from $r = a(1-\cos\theta)$ to $r = a\sin\theta$.

Then θ varies from $\theta = 0$ to $\theta = \pi/2$

$\therefore A \;= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,drd\theta$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \sin^2\theta - (1-\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \left(\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= a^2 \int^{\pi/2}$$

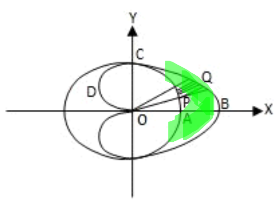
$$= \frac{a^2}{2}\int_0^{\pi/2} (-1 + 2\cos\theta - \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = \frac{a^2(4-\pi)}{4}$$

12. Find the area outside the circle $r = a$ and inside the cardioide $r = a\,(1 + cos\theta\,)$.

Solution: The circle $r = a$ and the cardioide $r = a(1 + \cos\theta)$ are as shown in the figure.



$r = a$
centre $\to (0,0)$
radius $\to a$

$r \to$ varies circle to cardioide
$a \to a(1+\cos\theta)$

When they intersect $a = a(1 + \cos\theta)$ $\therefore 1 = 1 + \cos\theta$ $\therefore \cos\theta = 0$ $\therefore \theta = \pm\pi/2$

$\therefore$ Area $= 2$ area $ABQCPA$

$$= 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\, dr d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2} \left[(1+\cos\theta)^2 - 1\right] d\theta$$

$$= a^2\int_0^{\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\, d\theta$$

$$= a^2\int_0^{\pi/2} \left[2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= a^2 \int_0 \left[2\cos\theta + \left(\frac{1+\cos 2\theta}{2} \right) \right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (1 + 4\cos\theta + \cos 2\theta)\, d\theta$$

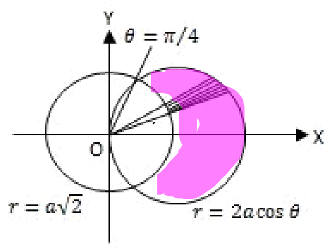
$$= \frac{a^2}{2} \left[\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{2} + 4 \right] = \frac{a^2}{4} [\pi + 8]$$

13. Find the area outside the circle $r = a\sqrt{2}$ and inside circle $r = 2a\, cos\theta$.

Solution: First we note that $r = a\sqrt{2}$ i.e. $r^2 = 2a^2$ i.e. $x^2 + y^2 = 2a^2$ is a circle with centre at the origin and radius $= a\sqrt{2}$ and $r = 2a\cos\theta$ i.e. $r^2 = 2ar\cos\theta$ i.e. $x^2 + y^2 = 2ax$ i.e. $(x-a)^2 + y^2 = a^2$ is the circle with centre at $(a, 0)$ and radius $= a$

$r \rightarrow a\sqrt{2}$ to $2a\cos\theta$



To find the point of intersection, we solve the two equations

$\therefore a\sqrt{2} = 2a\cos\theta$ i.e. $\cos\theta = \pm 1/\sqrt{2}, \therefore \theta = \pm\pi/4$

Area of the crescent $= 2\int_0^{\pi/4} \int_{a\sqrt{2}}^{2a\cos\theta} r\, dr d\theta$

$$= 2\int_0^{\pi/4} \left(\frac{r^2}{2} \right)_{a\sqrt{2}}^{2a\cos\theta} d\theta = \int_0^{\pi/4} (4a^2\cos^2\theta - 2a^2)\, d\theta$$

$$= 4a^2 \int_0^{\pi/4} \cos^2\theta\, d\theta - 2a^2 \int_0^{\pi/4} d\theta$$

$$= 4a^2 \int_0^{\pi/4} \left(\frac{1+\cos 2\theta}{2} \right) d\theta - 2a^2 (\theta)_0^{\pi/4}$$

$$\int_0 \left(\frac{\qquad}{2} \right) d\theta \quad - \quad (\;)_0$$

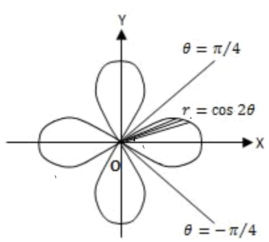
$$= 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4} - \frac{a^2\pi}{2}$$

$$= 2a^2 \left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - 0 \right] - \frac{a^2\pi}{2}$$

$$= a^2$$

14. Find by double integration the area enclosed by one loop of $r = \cos 2\theta$

Solution: The curve $r = \cos 2\theta$ is a four leaved rose.



complete area

$$= 4 \cdot \frac{\pi}{8} = \frac{\pi}{2}.$$

The area of one loop above the x −axis is $= \int_0^{\pi/4} \int_{r=0}^{\cos 2\theta} r dr d\theta$

The area of one loop is twice of this

$\therefore A \quad = 2 \int_{\theta=0}^{\pi/4} \int_{r=0}^{\cos 2\theta} r dr d\theta$

$$= 2 \int_0^{\pi/4} \left(\frac{r^2}{2} \right)_0^{\cos 2\theta} d\theta = \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1 + \cos 4\theta}{2} \right) d\theta = \frac{1}{2} \left[\theta + \frac{\sin 4\theta}{4} \right]_0^{\pi/4}$$

$$= \frac{1}{2} \left[\frac{\pi}{4} + 0 \right]$$

$$A = \frac{\pi}{8}$$