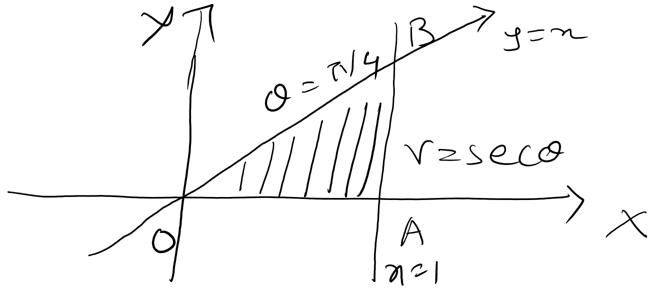
TYPE 5: TRANSFORMATION FROM CARTESIAN TO POLAR COORDINATES

Change to polar coordinates and evaluate.

1. $\int_0^1 \int_0^x (x+y)\,dy\,dx$

The region is bounded by the line $y=0$ ie the x-axis and the line $y=x$, the line $x=0$ ie the y-axis and the line $x=1$



To change the coordinate system,

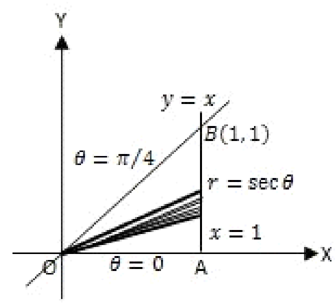
we put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

The line $y=x$ will become $r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

and the line $x=1$ becomes $r\cos\theta = 1 \Rightarrow r = \sec\theta$

Now this strip, r varies from 0 to $\sec\theta$

and θ varies from 0 to $\frac{\pi}{4}$



$$I = \int_0^{\pi/4} \int_0^{\sec\theta} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} (\cos\theta + \sin\theta)\,d\theta \int_0^{\sec\theta} r^2\,dr$$

$$= \int^{\pi/4} (\cos\theta + \sin\theta)\,d\theta \cdot \left(\frac{r^3}{3}\right)_0^{\sec\theta}$$

$$= \int_0 (\cos\theta + \sin\theta)\, d\theta \cdot \left(\frac{r^3}{3}\right)_0^{\sec\theta}$$

$$= \frac{1}{3}\int_0^{\pi/4} \left(\sec^2\theta + \frac{\sin\theta}{\cos^3\theta}\right) d\theta$$

put $\cos\theta = t$

$$= \frac{1}{3}\left[\int_0^{\pi/4} \sec^2\theta\, d\theta + \int_1^{1/\sqrt{2}} \frac{1}{t^3}(-dt)\right]$$

$$I = \frac{1}{3}\left[(\tan\theta)_0^{\pi/4} + \left(\frac{t^{-2}}{2}\right)_1^{1/\sqrt{2}}\right]$$

$$= \frac{1}{3}\left[(1+0) + \frac{1}{2}\left[2-1\right]\right]$$

$$I = \frac{1}{2}$$

2. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$

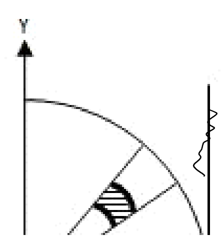
The region of integration is bounded by

$x = 0$ ie the y-axis

$x = \sqrt{1-y^2} \Rightarrow x^2+y^2 = 1$ ie the unit circle

$y = 0$ ie the x-axis

$y = 1$ a line parallel to x-axis



we change the coordinate system

$x = r\cos\theta$, $y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$

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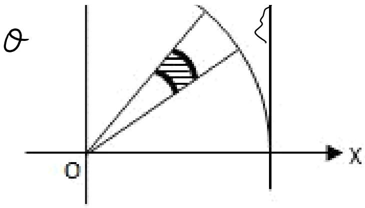


$x^2+y^2=1 \rightarrow r^2=1 \rightarrow r=1$

limits for r are 0 to 1

and θ varies from 0 to $\frac{\pi}{2}$ (as we have the first quadrant)

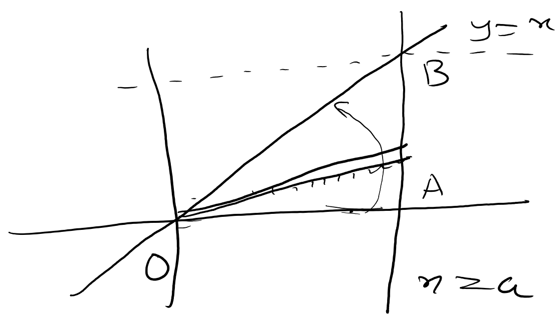
$$I = \int_0^{\pi/2}\int_0^1 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^4}{4}\right)_0^1 d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{1}{4}(\theta)_0^{\pi/2} = \frac{\pi}{8}$$

3. $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$

The region of integration is $x=y$, $x=a$, $y=0$, $y=a$

ie the triangle OAB



Putting $x = r\cos\theta$, $y = r\sin\theta$

and $dx\,dy = r\,dr\,d\theta$

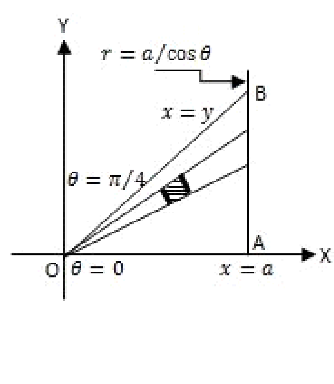
the line $x=a$ becomes

$r\cos\theta = a \Rightarrow r = a\sec\theta$

then line $y=x$ becomes $r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

$\therefore r$ varies from 0 to $a\sec\theta$

θ varies from 0 to $\frac{\pi}{4}$



$$I = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{r} \cdot r\,dr\,d\theta$$

$$I = \int_0 \int_0 \frac{r^2\cos^2\theta}{r} \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \cdot \left(\frac{r^3}{3}\right)_0^{a\sec\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta \cdot a^3\sec^3\theta \, d\theta \quad = \frac{a^3}{3}\int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \frac{a^3}{3}\Big[\log(\sec\theta + \tan\theta)\Big]_0^{\pi/4}$$

$$= \frac{a^3}{3}\Big[\log(\sqrt{2}+1) - \log(1)\Big]$$

$$I = \frac{a^3}{3}\log(1+\sqrt{2})$$

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4. $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2-y^2}} dy dx$

The limits of y are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$

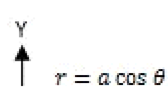
$y = \sqrt{ax-x^2} \Rightarrow x^2+y^2 = ax \Rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$

circle with centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

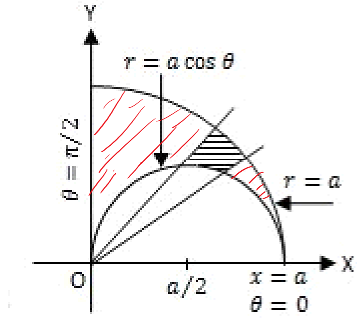
$y = \sqrt{a^2-x^2} \Rightarrow x^2+y^2 = a^2$

Circle with centre at $(0,0)$ and radius a.

upper halves of both the



upper halves of both the circles



To change the given integral to polar

we put $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

$$x^2+y^2 = ax \Rightarrow r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$$

$$x^2+y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$$

Hence the limit for r are $r = a\cos\theta$ to $r = a$ and limit for θ will be 0 to $\frac{\pi}{2}$

$$I = \int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$a^2 - r^2 = t$$

$$-2r\,dr = dt$$

$$r = a\cos\theta \qquad t = a^2(1-\cos^2\theta)$$

$$r = a \qquad t = 0$$

$$I = \int_0^{\pi/2}\int_{a^2(1-\cos^2\theta)}^{0} \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\right)dt\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^{a^2\sin^2\theta} \frac{1}{\sqrt{t}}\,dt\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\frac{t^{1/2}}{1/2}\right]_0^{a^2\sin^2\theta} d\theta = \int_0^{\pi/2} a\sin\theta\,d\theta$$

$$I = a\left[-\cos\theta\right]_0^{\pi/2} = a$$

5. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} log\left(x^2+y^2\right) dx\, dy$

The region is bounded by

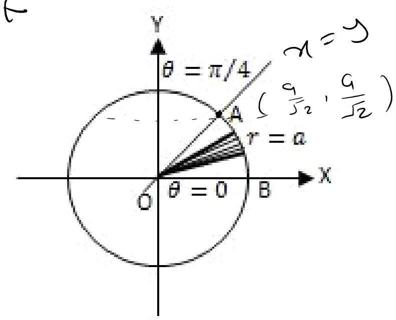
$x = y$, the line through origin

$x = \sqrt{a^2-y^2} \Rightarrow x^2+y^2 = a^2$; a circle with centre at origin and radius a.

$y = 0$: the x-axis

$y = a/\sqrt{2}$: a line parallel to x-axis

The line and the circle intersect in $A\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$



To change to polar, we put

$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

$x = y$ will change to $r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

$x^2+y^2 = a^2 \Rightarrow r = a$

The region of integration is OAB

on this strip r varies from 0 to a

θ varies from 0 to $\pi/4$

$$I = \int_0^{\pi/4}\int_0^{a} \log(r^2)\, r\,dr\,d\theta$$

$\pi/4 \quad a$

$$= \int_0^{\pi/4}\int_0^{a} 2(\log r)\cdot r\, dr\, d\theta$$

$$= 2\int_0^{\pi/4}\left[(\log r)\left(\frac{r^2}{2}\right) - \int \frac{1}{r}\cdot\frac{r^2}{2}\, dr\right]_0^a d\theta$$

$$= 2\int_0^{\pi/4}\left[(\log r)\left(\frac{r^2}{2}\right) - \frac{r^2}{4}\right]_0^a d\theta$$

$$= 2\int_0^{\pi/4}\left(\frac{a^2}{2}\log a - \frac{a^2}{4}\right) d\theta$$

$$= 2\left(\frac{a^2}{2}\log a - \frac{a^2}{4}\right)\left(\frac{\pi}{4}\right) = a^2\left(\log a - \frac{1}{2}\right)\cdot\frac{\pi}{4}$$

6. $\int_0^{4a}\int_{y^2/4a}^{y}\left(\frac{x^2-y^2}{x^2+y^2}\right) dx\, dy$

The limits of x are $x = \frac{y^2}{4a}$ ie $y^2 = 4ax \rightarrow$ parabola.

the line $x = y$: passing through origin

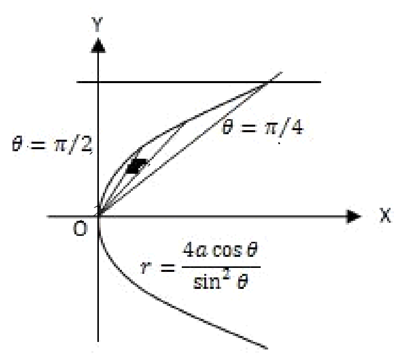
limits for y are $y = 0$: the x-axis

$y = 4a$: a line parallel to x axis

By changing to polar

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.



$y^2 = 4ax$

$r^2\sin^2\theta = 4ar\cos\theta$

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

The line $y = x$ changes to $\theta = \frac{\pi}{4}$

Consider a strip in the region of integration

r varies from 0 to $\frac{4a\cos\theta}{\sin^2\theta}$

and θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$$I = \int_{\pi/4}^{\pi/2} \int_{0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} \, r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \int_{0}^{\frac{4a\cos\theta}{\sin^2\theta}} (\cos^2\theta - \sin^2\theta) \, r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta) \left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta) \cdot \frac{\cos^2\theta}{\sin^4\theta} \, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta) \, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^2\theta\,\mathrm{cosec}^2\theta - 2\,\mathrm{cosec}^2\theta + 2) \, d\theta$$

$$= 8a^2 \left[-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta \right]^{\pi/2}$$

$$= 8a^2\left[-\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right]_{\pi/4}^{\pi/2}$$

$$I = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

7. $\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\, dy\, dx$

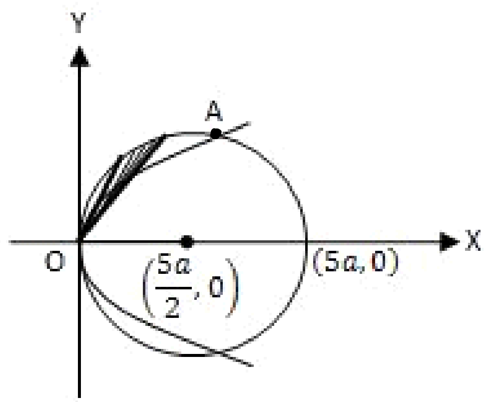
The region is bounded by

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax \rightarrow$ parabola.

and $y = \sqrt{5ax - x^2} \Rightarrow x^2 + y^2 = 5ax$

$$\Rightarrow \left(x - \frac{5a}{2}\right)^2 + y^2 = \left(\frac{5a}{2}\right)^2$$

This is a circle with centre at $\left(\frac{5a}{2}, 0\right)$ and radius $\frac{5a}{2}$



The point of intersection of the circle and the parabola is $(a, 2a)$

To change to polar, we put

$x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

The parabola $y^2 = 4ax$ becomes $r^2\sin^2\theta = 4ar\cos\theta$

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

and the circle $x^2 + y^2 = 5ax \Rightarrow r^2 = 5ar\cos\theta$

$$\Rightarrow r = 5a\cos\theta$$

Now at $A(a, 2a)$, $x = r\cos\theta = a$

$y = r\sin\theta = 2a$

$\Rightarrow \tan\theta = 2 \quad \Rightarrow \theta = \tan^{-1} 2$

$$\therefore I = \int_{\tan^{-1}2}^{\pi/2} \int_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta} \frac{r}{r^2\sin^2\theta} \cdot r\,dr\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2} \int_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta} \frac{1}{\sin^2\theta}\,dr\,d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2} \frac{1}{\sin^2\theta} \left[r\right]_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta} d\theta$$

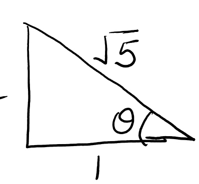
$$= \int_{\tan^{-1}2}^{\pi/2} \frac{1}{\sin^2\theta} \left[5a\cos\theta - \frac{4a\cos\theta}{\sin^2\theta}\right] d\theta$$

$$= \int_{\tan^{-1}2}^{\pi/2} \left(\frac{5a}{\sin^2\theta} - \frac{4a}{\sin^4\theta}\right) \cos\theta\,d\theta$$

put $\sin\theta = t \rightarrow \cos\theta\,d\theta = dt$

when $\theta = \tan^{-1} 2 \longrightarrow t = \frac{2}{\sqrt{5}}$

$\theta = \frac{\pi}{2} \quad t = 1$



$$\therefore I = 5a \int^{1} \frac{dt}{t^2} - 4a \int^{1} \frac{dt}{t^4}$$

$$\therefore I = 5a\int_{2/\sqrt{5}}^{1} \frac{dt}{t^2} - 4a\int_{2/\sqrt{5}}^{1} \frac{dt}{t^4}$$

$$= 5a\left[\frac{-1}{t}\right]_{\frac{2}{\sqrt{5}}}^{1} + \frac{4a}{3}\left[\frac{1}{t^3}\right]_{\frac{2}{\sqrt{5}}}^{1}$$

$$I = \frac{a}{3}\left[5\sqrt{5} - 11\right]$$

8. $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dy\, dx$

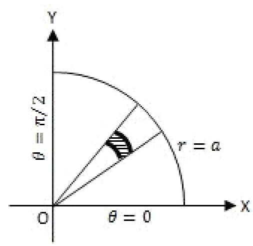
The region of integration is bounded by $y=0$, $y=\sqrt{a^2-x^2}$, $x=0$ and $x=a$

ie x-axis, circle $x^2+y^2=a^2$, y axis and $x=a$

To change to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



$x^2+y^2=a^2 \longrightarrow r = a$

r limit from 0 to a

θ limit to 0 to $\pi/2$

$$I = \int_0^{\pi/2}\int_0^{a} e^{-r^2}\, r\, dr\, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{a} e^{-r^2}\cdot r\, dr\right)$$

$$= \left(\int_0 d\theta\right)\left(\int_0 e^{-r^2}\cdot r\,dr\right)$$

$$= \frac{\pi}{4}\left(1-e^{-a^2}\right)$$

H.W

9. $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$

10. $\int_0^a \int_y^{a+\sqrt{a^2-y^2}} \frac{1}{(4a^2+x^2+y^2)^2}\,dydx$

The curve $x = a+\sqrt{a^2-y^2} \Rightarrow (x-a) = \sqrt{a^2-y^2}$

$\Rightarrow (x-a)^2+y^2 = a^2$

circle with centre at $(a,0)$ and radius a.

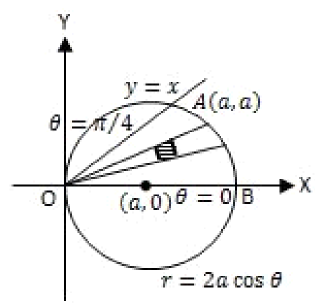
$x=y$ line passing through origin

The region is OAB.

To change to polar coordinates

$x = r\cos\theta, \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



$(x-a)^2+y^2 = a^2$

$x^2+y^2 = 2ax$

$r^2 = 2ar\cos\theta \quad \rightarrow r = 2a\cos\theta$

The line $y=x$ will change to $\theta = \frac{\pi}{4}$.

In the strip, r varies from 0 to $2a\cos\theta$

θ varies from 0 to $\frac{\pi}{4}$

$$I = \int^{\pi/4} \int^{2a\cos\theta} \frac{1}{(4a^2+r^2)^2}\, r\,dr\,d\theta$$

$$I = \int_0 \int_0 \frac{1}{(4a^2+r^2)^2}\, r\,dr\,d\theta$$

put $4a^2+r^2 = t$ $\quad 2r\,dr = dt$

$r=0 \Rightarrow t = 4a^2$

$r = 2a\cos\theta\,,\ t = 4a^2(1+\cos^2\theta)$

$$I = \int_0^{\pi/4}\int_{4a^2}^{4a^2(1+\cos^2\theta)} \frac{1}{t^2}\cdot\frac{dt}{2}\,d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/4}\left(\frac{1}{t}\right)_{4a^2}^{4a^2(1+\cos^2\theta)} d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4}\left[1-\frac{1}{1+\cos^2\theta}\right]d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4}\left(1-\frac{\sec^2\theta}{\sec^2\theta+1}\right)d\theta$$

$$= \frac{1}{8a^2}\left[\int_0^{\pi/4}d\theta - \int_0^1 \frac{dt}{2+t^2}\,dt\right] \quad \left[\text{put } t=\tan\theta\right]$$

$$= \frac{1}{8a^2}\left[(\theta)_0^{\pi/4} - \left(\frac{1}{\sqrt2}\tan^{-1}\frac{t}{\sqrt2}\right)_0^1\right]$$

$$= \frac{1}{8a^2}\left[\frac{\pi}{4} - \frac{1}{\sqrt2}\tan^{-1}\left(\frac{1}{\sqrt2}\right)\right]$$

11. $\int_0^a \int_y^a x\,dxdy$

The region of integration is bounded by y

11. $\int_0 \int_y x\,dx\,dy$

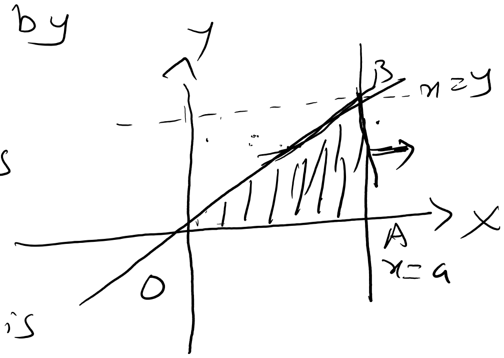
The region of integration is bounded by

$x = y$: line passing thru origin

$x = a$: line parallel to y-axis

$y = 0$: the x-axis

$y = a$: line parallel to x-axis



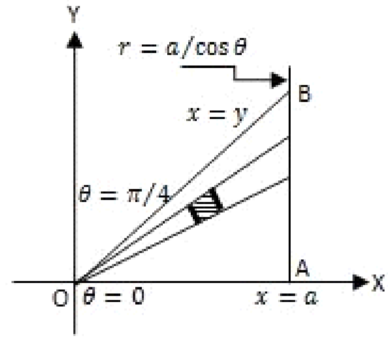
put $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$

$x = y$ will change to $\theta = \frac{\pi}{4}$

$x = a$ will become

$$r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$$



In this strip,

r varies from 0 to $r = \frac{a}{\cos\theta}$

θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \int_0^{a/\cos\theta} r\cos\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta \cdot \left(\frac{r^3}{3}\right)_0^{a/\cos\theta} d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec^2\theta\,d\theta = \frac{a^3}{3}\left(\tan\theta\right)_0^{\pi/4} = \frac{a^3}{3}$$

H.W.

12. $\int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} \frac{xdydx}{\sqrt{(x^2+y^2)}}$

1. **12.** $\int_0^{a/\sqrt{2}}\int_x^{\sqrt{a^2-x^2}} \frac{xdydx}{\sqrt{(x^2+y^2)}}$

H.W **13.** $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dydx$

14. $\int_0^{4a}\int_{y^2/4a}^{y} dxdy$

The region of integration is bounded by

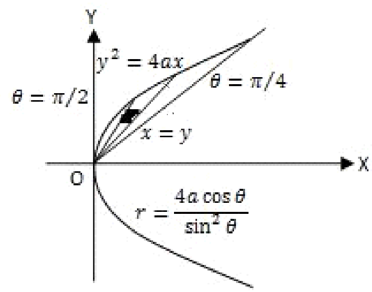
$x = \frac{y^2}{4a} \Rightarrow y^2 = 4ax \rightarrow$ parabola.

$x = y \rightarrow$ line passing through origin

By putting $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



parabola $y^2 = 4ax$ changes to $r = \frac{4a\cos\theta}{\sin^2\theta}$

The line $y = x$ changes to $\theta = \frac{\pi}{4}$.

In the strip, r varies from 0 to $\frac{4a\cos\theta}{\sin^2\theta}$

θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$

$$I = \int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{16a^2\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= \frac{1}{2} \int_{\pi/4} \frac{16 a^2 \cos^2\theta}{\sin^4\theta} \, d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta \ \mathrm{cosec}^2\theta \, d\theta$$

put $\cot\theta = t$

$-\mathrm{cosec}^2\theta \, d\theta = dt$

$\theta = \frac{\pi}{4}, \ t = 1 \ ; \theta = \frac{\pi}{2} \ t = 0$

$$= 8a^2 \int_1^0 t^2 (-dt) = 8a^2 \int_0^1 t^2 dt$$

$$= 8a^2 \left(\frac{t^3}{3} \right)_0^1 = \frac{8a^2}{3}.$$

TYPE 6: EVALUATION OF DOUBLE INTEGRALS OVER THE GIVEN REGIONS BY CHANGING TO POLAR COORDINATES

Evaluate the following integrals over the region stated, by changing to polar coordinates.

1. $\iint y^2 dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$

Sol^n:- First we note that

$x^2 + y^2 - ax = 0 \Rightarrow (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$

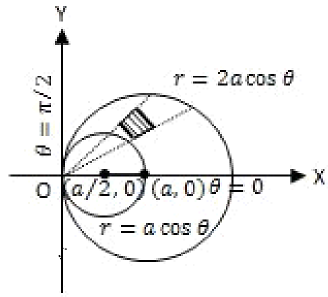
This is a circle with centre at $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$

Also $x^2 + y^2 - 2ax = 0 \Rightarrow (x-a)^2 + y^2 = a^2$

This is a circle with centre at $(a, 0)$ and radius a.

we change the given integral into polar coordinates put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.



$\therefore$ $x^2+y^2-ax=0$ will change to $r^2 - ar\cos\theta = 0$

$\Rightarrow r = a\cos\theta$

Also $x^2+y^2-2ax=0$ will change to $r^2-2ar\cos\theta=0$

$\Rightarrow r = 2a\cos\theta$

The region of integration is outside $r = a\cos\theta$ and inside $r = 2a\cos\theta$.

In this region, r varies from $a\cos\theta$ to $2a\cos\theta$

θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\therefore I = \int_{-\pi/2}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta \; r\,dr\,d\theta$$

$$= 2\int_{0}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} \sin^2\theta \; r^3\,dr\,d\theta$$

$$= 2\int_{0}^{\pi/2} \sin^2\theta \left(\frac{r^4}{4}\right)_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} \sin^2\theta \left[16a^4\cos^4\theta - a^4\cos^4\theta\right] d\theta$$

$$= \frac{15a^4}{2}\int_{0}^{\pi/2} \sin^2\theta\cos^4\theta\,d\theta$$

$$= \frac{15a^4}{2} \cdot \frac{1}{2} B\left(\frac{2+1}{2}, \frac{4+1}{2}\right) = \frac{15a^4}{4} B\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$= \frac{15a^4}{4} \cdot \frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{\Gamma 4}$$

$$I = \frac{15\pi a^4}{64}$$

2. $\iint \frac{(x^2+y^2)^2}{x^2y^2} dx\, dy$ over the area common to $x^2+y^2 = ax$ and $x^2+y^2 = by, a, b > 0$

Soln:- First we note that

$$x^2+y^2 = ax \rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

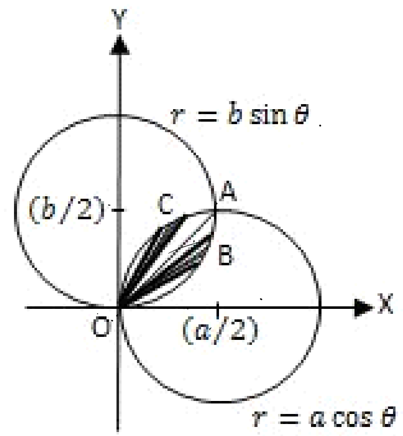
This is a circle with centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

Also $x^2+y^2 = by \rightarrow x^2 + \left(y-\frac{b}{2}\right)^2 = \frac{b^2}{4}$

This is a circle with centre at $\left(0, \frac{b}{2}\right)$ and radius $\frac{b}{2}$

Converting to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$

$$dx\,dy = r\,dr\,d\theta$$



$x^2+y^2-ax=0 \rightarrow r = a\cos\theta$

$x^2+y^2-by=0 \rightarrow r = b\sin\theta$

The region of integration is OBACO

The point of intersection A is given by

$r = a\cos\theta$, $r = b\sin\theta \Rightarrow a\cos\theta = b\sin\theta$

$$\Rightarrow \tan\theta = \frac{a}{b} \Rightarrow \theta = \tan^{-1}\left(\frac{a}{b}\right) = \alpha \text{ (say)}$$

so, our region is divided into two parts

in region OBA, r varies from 0 to $b\sin\theta$
θ varies from 0 to α

in the region OCA, r varies from 0 to $a\cos\theta$
θ varies from α to $\frac{\pi}{2}$

$$\therefore I = \int_0^{\alpha}\int_0^{b\sin\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta$$

$$+ \int_{\alpha}^{\pi/2}\int_0^{a\cos\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\, r\,dr\,d\theta$$

$$= \int_0^{\alpha} \frac{1}{\sin^2\theta\cos^2\theta}\cdot\left(\frac{r^2}{2}\right)_0^{b\sin\theta} d\theta + \int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta\cos^2\theta}\cdot\left(\frac{r^2}{2}\right)_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\alpha} \frac{1}{\sin^2\theta\cos^2\theta}\cdot b^2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\alpha}^{\pi/2} \frac{1}{\sin^2\theta\cos^2\theta}\cdot a^2\cos^2\theta\,d\theta$$

$$= \frac{b^2}{2}\int_0^{\alpha}\sec^2\theta\,d\theta + \frac{a^2}{2}\int_{\alpha}^{\pi/2}\operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{b^2}{2}\left(\tan\theta\right)_0^{\alpha} + \frac{a^2}{2}\left(-\cot\theta\right)_{\alpha}^{\pi/2}$$

$$= \frac{b^2}{2}\left(\tan\alpha - 0\right) - \frac{a^2}{2}\left(0 - \cot\alpha\right)$$

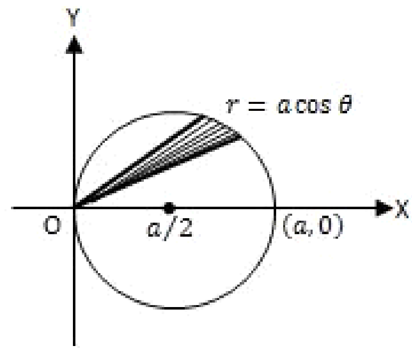
$$= \frac{b^2}{2}\left[\tan\left(\tan^{-1}\frac{a}{b}\right)\right] + \frac{a^2}{2}\cot\left(\tan^{-1}\left(\frac{a}{b}\right)\right)$$

$$= \frac{b^2}{2}\left(\frac{a}{b}\right) + \frac{a^2}{2}\left(\frac{b}{a}\right)$$

$\therefore I = ab.$

3. $\iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$ where R is the area of the upper half of the circle $x^2 + y^2 = ax$

The region of integration is upper half of circle $x^2+y^2 = ax$ which has centre at $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$



Converting to polar coordinates by putting $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = ax$ changes to $r = a\cos\theta$

In this strip r changes from 0 to $a\cos\theta$

θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\cdot r\,dr\,d\theta$$

put $a^2 - r^2 = t \rightarrow -2r\,dr = dt$

complete this sum as a Homework.

Ans:- $\frac{a^3}{18}(3\pi - 4)$

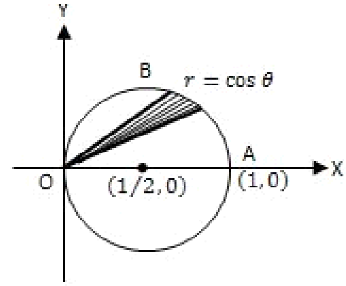
4. $\iint_R \frac{1}{\sqrt{xy}}\, dxdy$ where R is the region of integration bounded by $x^2 + y^2 - x = 0$ and $y \geq 0$

The region is given by circle $x^2+y^2-x = 0$ in the first quadrant

Converting to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,dr\,d\theta$

$x^2+y^2-x=0$ changes to $r=\cos\theta$



$\therefore$ In this region,

r varies from 0 to $\cos\theta$

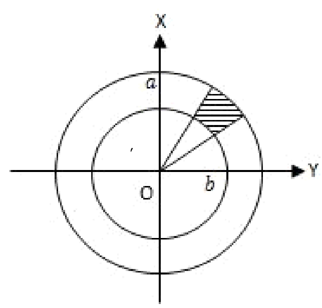
θ varies from 0 to $\frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2}\int_0^{\cos\theta} \frac{1}{\sqrt{r^2\cos\theta\sin\theta}} \cdot r\,dr\,d\theta$$

Evaluation is Home work. Ans :- $\frac{\pi}{\sqrt{2}}$

5. $\iint \frac{x^2y^2}{x^2+y^2}$ over the annular region between circles $x^2+y^2=a^2$ and $x^2+y^2=b^2; a>b(a,b>0)$

Both the circle have centre at origin and the region of integration is the area betn both the circle.



Changing to polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\,dr\,d\theta$

$x^2+y^2=a^2 \longrightarrow r^2=a^2 \longrightarrow r=a$

$x^2+y^2=b^2 \longrightarrow r^2=b^2 \longrightarrow r=b$

In this region, r varies from b to a

θ varies from 0 to 2π

$$\therefore I = \int_0^{2\pi}\int_b^a \frac{r^4 \sin^2\theta \cos^2\theta}{r^2}\, r\,dr\,d\theta$$

$$= \left(4\int_0^{\pi/2} \sin^2\theta \cdot \cos^2\theta\, d\theta\right)\left(\int_b^a r^3\,dr\right)$$

$$= 4\cdot\frac{1}{2} B\left(\frac{3}{2},\frac{3}{2}\right)\left(\frac{r^4}{4}\right)_b^a$$

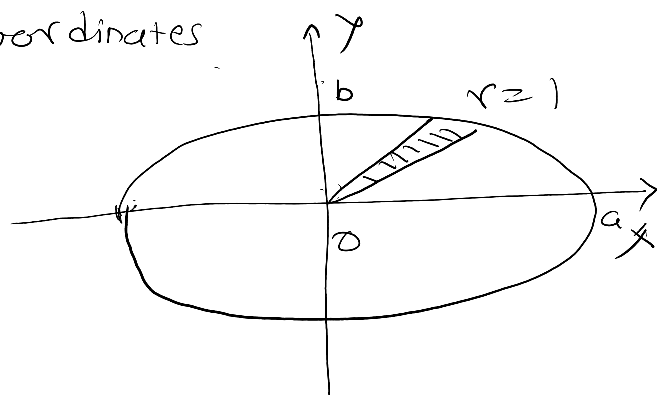
$$I = (a^4 - b^4)\cdot\frac{\pi}{16}$$

6. $\iint \sqrt{\left[\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}\right]}\,dx\,dy$ where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Soln:- we use elliptical polar coordinates

$x = ar\cos\theta,\quad y = br\sin\theta$

$dx\,dy = ab\,r\,dr\,d\theta$



The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Changes to $r^2=1 \Rightarrow r=1$

The limits for r will be 0 to 1

θ will be 0 to 2π

$$\therefore I = \int^{2\pi}\int^{1} \sqrt{\frac{a^2b^2(1-r^2)}{a^2b^2(1+r^2)}}\cdot ab\,r\,dr\,d\theta$$

$$\therefore I = \int_0 \int_0 \sqrt{\frac{a^2b^2(1-r^2)}{a^2b^2(1+r^2)}} \cdot ab\,r\,dr\,d\theta$$

$$= 4ab \int_0^{\pi/2} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr\,d\theta$$

$$= 4ab \int_0^{\pi/2} \int_0^1 \frac{1-r^2}{\sqrt{1-r^4}} \cdot r\,dr\,d\theta$$

put $r^2 = \sin t \rightarrow 2r\,dr = \cos t\,dt$

when $r=0 \quad t=0$

$r=1 \quad t=\pi/2$

$$= 4ab \int_0^{\pi/2} \int_0^{\pi/2} \frac{1-\sin t}{\cos t} \cdot \frac{1}{2}\cos t\,dt\,d\theta$$

$$= 2ab \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} (1-\sin t)\,dt\right)$$

$$= 2ab\left(\frac{\pi}{2}\right)\left(t + \cos t\right)_0^{\pi/2}$$

$$= \pi ab\left(\frac{\pi}{2} + 0 - 0 - 1\right)$$

$$I = \pi ab\left(\frac{\pi}{2} - 1\right)$$

7. $\iint \frac{1}{(1+x^2+y^2)^2}\,dxdy$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$

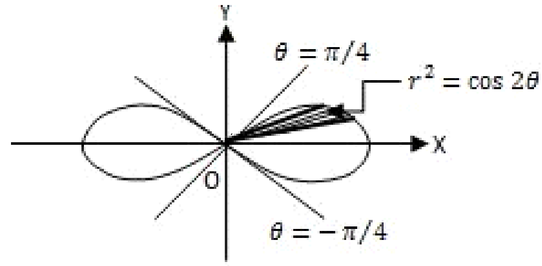
Changing to polar coordinates

$(1+x^2+y^2)$

Changing to polar coordinates

by putting $x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$



$(x^2+y^2)^2 = x^2 - y^2$ becomes

$$r^4 = r^2(\cos^2\theta - \sin^2\theta)$$

$$r^2 = \cos 2\theta$$

$$r = \sqrt{\cos 2\theta}$$

$\therefore$ In this region r varies from 0 to $\sqrt{\cos 2\theta}$

θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

$$I = \int_{-\pi/4}^{\pi/4} \int_{0}^{\sqrt{\cos 2\theta}} \frac{1}{(1+r^2)^2}\, r\,dr\,d\theta$$

put $1 + r^2 = t$

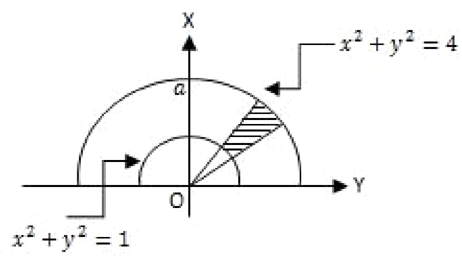
Evaluation part is H.W. $\underline{\text{Ans}}$ $\frac{\pi}{4} - \frac{1}{2}$

8. Evaluate $\iint_R (3x + 4y^2)\,dxdy$ where R is the region in the upper half of the area bounded by the circle $x^2 + y^2 = 1, x^2 + y^2 = 4$

By changing to polar coordinates

$x^2 + y^2 = 1 \longrightarrow r = 1$

$x^2 + y^2 = 4 \longrightarrow r = 2$



$$\therefore I = \int_0^{\pi} \int_1^{2} (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int^{\pi} \int^{2}$$

$$= \int_0^{\pi} \int_1^{2} \left(3\cos\theta \cdot r^2 + 4\sin^2\theta \cdot r^3\right) dr\, d\theta$$

$$= \int_0^{\pi} \cos\theta \cdot (r^3)_1^2 + \sin^2\theta \, (r^4)_1^2 \, d\theta$$

$$= \int_0^{\pi} 7\cos\theta + 15\sin^2\theta \, d\theta$$

$$= \int_0^{\pi} 7\cos\theta + 15\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= 7(\sin\theta)_0^{\pi} + \frac{15}{2}(\theta)_0^{\pi} - \frac{15}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$I = \frac{15\pi}{2}$$

9. Evaluate $\iint_R x^3 y \, dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

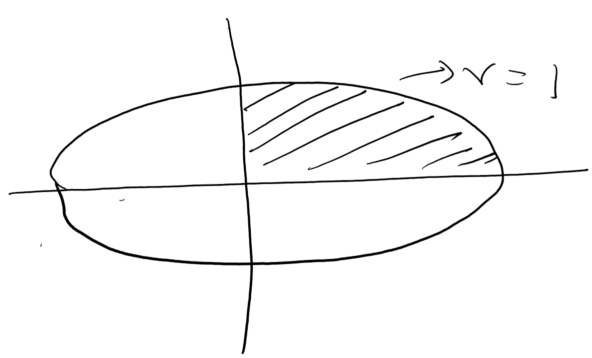
$x = ar\cos\theta$

$y = br\sin\theta$

$dxdy = abr\, dr\, d\theta$

$r \rightarrow 0$ to 1

$\theta \rightarrow 0$ to $\frac{\pi}{2}$



$\pi/2 \quad 1$

θ : 0 to $\frac{\pi}{2}$

$$I = \int_0^{\pi/2} \int_0^1 a^3 r^3 \cos^3\theta \cdot br \sin\theta \; abr \, dr \, d\theta$$

$$= a^4 b^2 \left[\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta \right] \left[\int_0^1 r^5 dr \right]$$

$$= a^4 b^2 \cdot \frac{1}{2} B(2, 1) \left(\frac{r^6}{6} \right)_0^1$$

Area

$$= \frac{a^4 b^2}{24}$$