EVALUATION IN POLAR COORDINATES

Monday, May 17, 2021 11:30 AM

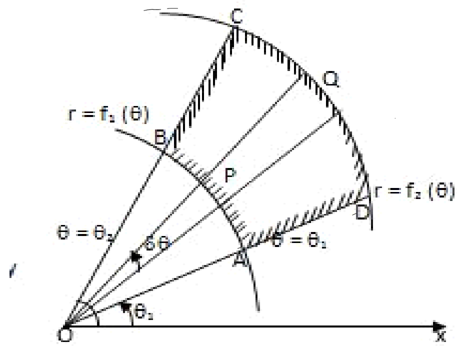
DOUBLE INTEGRALS IN POLAR COORDINATES:

If the region of integration is a circle, ellipse, lemniscate, etc. or if the integral contains an expression like $\left(x^2+y^2\right)^{n/2}$ then it is easier to solve the problem by transforming into polar by putting

$x = r\cos\theta, y = r\sin\theta,\ \ dx\,dy = r\,d\theta\,dr$ (in circle)

or $x = ar\cos\theta, y = br\sin\theta, dx\,dy = abr\,d\theta\,dr$ (in ellipse)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Here consider the region ABCD bounded by the polar curves $r = f_1(\theta), r = f_2(\theta)$ and the lines $\theta = \theta_1 and\ \theta = \theta_2$.

The strips are always radial w.r.t the pole.

Here the radial elementary strip (called a wedge) of angular thickness $r\delta\theta$ such as PQ extends from $r = f_1(\theta), at\ P\ to\ r = f_2(\theta)\ at\ Q$

To Cover the entire region of integration, then such strip slides from $\theta = \theta_1\ to\ \theta = \theta_2$.

Hence $I = \int_{\theta_1}^{\theta_2}\int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)\, r d\theta\, dr$

THEOREM: Prove that $B(m,n) = \frac{\overline{|m}\,\overline{|n}}{|m+n}$

Proof: To prove the property we assume a property of double integral.

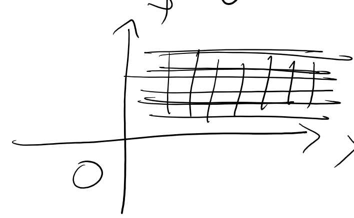
"If limits of both the integrals are constants and if variables can be separated then the given double integral can be expressed as a product of two integrals as follows:

$\int_a^b\int_c^d f(x,y)dx\,dy = \int_a^b\int_c^d g(x).\emptyset(y)dx\,dy = \left[\int_a^b g(x)dx\right]\left[\int_c^d \emptyset(y)dy\right]$"

$\overline{|m} = 2\int_0^\infty e^{-x^2}x^{2m-1}dx$

Now consider second form of Gamma function, We have

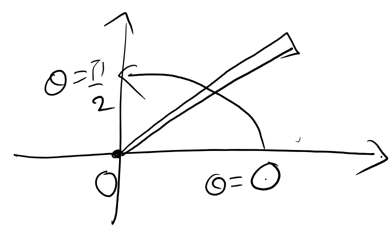
$$\overline{|m}\,.\overline{|n} = \left(2\int_0^\infty e^{-x^2}x^{2m-1}dx\right)\left(2\int_0^\infty e^{-y^2}y^{2n-1}dy\right)$$
$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dx\,dy$$



We now change the integral on r.h.s to polar form by putting $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = rd\theta\,dr$.

Since $x\ and\ y$ change from $0\ to\ \infty$ the region is the entire first quadrant.

To span this region in polar coordinates r must vary from $0\ to\ \infty$ and θ from 0 to $\frac{\pi}{2}$



$$\therefore\ \overline{|m}\,.\overline{|n} = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r^{2m-1}(\cos\theta)^{2m-1}\,.r^{2n-1}(\sin\theta)^{2n-1}\cdot r\,d\,r\,d\,\theta$$
$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r^{2(m+n)-1}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}d\,r\,d\theta$$
$$= \left(2\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,d\,r\right)\cdot\left(2\int_0^{\pi/2}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}\,d\,\theta\right)$$
$$= \overline{|m+n}\,B(m,n)$$

$r \to 0$ to ∞

$\theta \to 0$ to $\frac{\pi}{2}$

As the first integral is a Gamma function and the second integral is a Beta Function.

$$\therefore B(m,n) = \frac{\overline{|m}\,\overline{|n}}{|m+n}$$

TYPE 4: EVALUATION OF DOUBLE INTEGRALS IN POLAR COORDINATES

Evaluate the following integrals

1. $\iint r^3\, dr d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Sol: The circle $r = 2\sin\theta$

$$r^2 = 2r\sin\theta \quad \rightarrow x^2+y^2-2y=0$$

$$x^2+y^2 = 2y \quad \rightarrow x^2+(y-1)^2 = 1$$

circle with centre at $(0,1)$ and radius 1

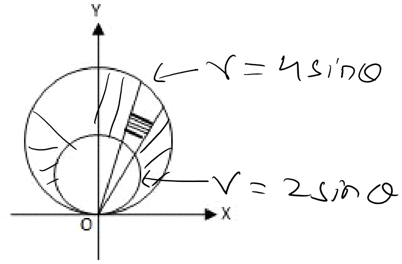
$$r = 4\sin\theta \rightarrow x^2+(y-2)^2 = 4$$

circle with centre at $(0,2)$ and radius 2

In this region

r varies from $2\sin\theta$ to $4\sin\theta$

and then θ varies from 0 to π



$$I = \int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\pi} \left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} 4^4\sin^4\theta - 2^4\sin^4\theta\, d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta = 120\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$\left[\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx\right.$$

if $f(2a-x) = f(x)]$

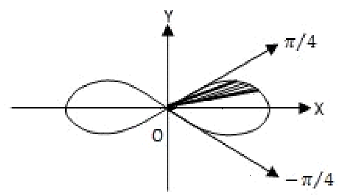
$$I = 120 \cdot \frac{1}{2} B\left(\frac{4+1}{2}, \frac{0+1}{2}\right) = 60\, B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$I = \frac{45\pi}{2}$$

2. Evaluate $\iint \frac{r\,dr d\theta}{\sqrt{r^2+a^2}}$ over one loop of the lemniscate $r^2 = \cancel{a^2 \cos\theta}$ $a^2 \cos 2\theta$

In the given region

r varies from 0 to $r = a\sqrt{\cos 2\theta}$



and θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

$$I = \int_{-\frac{\pi}{4}}^{\pi/4} \int_{0}^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{r^2+a^2}}\, dr\, d\theta$$

put $r^2 + a^2 = t$

$2r\,dr = dt$

$r = 0$; $t = a^2$

$r = a\sqrt{\cos 2\theta}$; $t = a^2(1+\cos 2\theta)$

$$I = \int_{-\frac{\pi}{4}}^{\pi/4} \int_{a^2}^{a^2(1+\cos 2\theta)} \frac{1}{\sqrt{t}} \cdot \frac{1}{2}\, dt \cdot d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\pi/4} \cdot \frac{1}{2} \cdot \left(\frac{\sqrt{t}}{1/2}\right)_{a^2}^{a^2(1+\cos 2\theta)} d\theta$$

$$= \int^{\pi/4} \left(a\sqrt{1+\cos 2\theta} - a\right) d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \left(a\sqrt{1+\cos 2\theta} - a\right) d\theta$$

$$= a\int_{-\pi/4}^{\pi/4} \left(\sqrt{2}\cos\theta - 1\right) d\theta$$

$$= a\left[\sqrt{2}\sin\theta - \theta\right]_{-\frac{\pi}{4}}^{\pi/4}$$

$$I = \left(2 - \frac{\pi}{2}\right) a$$

3. Evaluate $\iint re^{-r^2/a^2}\cos\theta\sin\theta\, d\theta dr$ over the upper half of the circle $r = 2a\cos\theta$

Soln: $r = 2a\cos\theta \Rightarrow r^2 = 2ar\cos\theta$

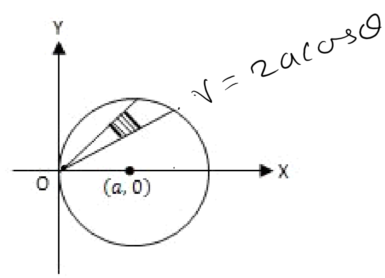
$$\Rightarrow x^2 + y^2 = 2ax$$

$$\Rightarrow x^2 - 2ax + y^2 = 0$$

$$\Rightarrow (x-a)^2 + y^2 = a^2$$

This is a circle with centre at $(a,0)$ and radius a.

In this region, r varies from 0 to $2a\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$



$$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} re^{-r^2/a^2}\sin\theta\cos\theta\, dr\, d\theta$$

put $\frac{r^2}{a^2} = t \Rightarrow r^2 = a^2 t$

$2r\,dr = a^2\,dt$

when $r=0$; $t=0$

$r = 2a\cos\theta$; $t = 4\cos^2\theta$

$$J = \int_0^{\pi/2}\int_0^{4\cos^2\theta} e^{-t}\,\frac{a^2}{2}\,dt \quad \sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{a^2}{2}\left(-e^{-t}\right)_0^{4\cos^2\theta} \sin\theta\cos\theta\,d\theta$$

$$= -\frac{a^2}{2}\int_0^{\pi/2}\left(e^{-4\cos^2\theta} - 1\right)\sin\theta\cos\theta\,d\theta$$

$$= -\frac{a^2}{2}\left[\int_0^{\pi/2} e^{-4\cos^2\theta}\sin\theta\cos\theta\,d\theta - \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\right]$$

put $-\cos^2\theta = t$

$2\sin\theta\cos\theta\,d\theta = dt$

$\theta = 0$; $t = -1$

$\theta = \pi/2$; $t = 0$

$$J = -\frac{a^2}{2}\left[\int_{-1}^{0} e^{4t}\cdot\frac{dt}{2} - \frac{1}{2}B\left(\frac{1+1}{2}, \frac{1+1}{2}\right)\right]$$

$$= -\frac{a^2}{4}\left(\frac{e^{4t}}{4}\right)_{-1}^{0} + \frac{a^2}{4}B(1,1)$$

$$= -\frac{a^2}{4}\left(\frac{1}{4} - \frac{e^{-4}}{4}\right) + \frac{a^2}{4}\,\frac{\Gamma 1\,\Gamma 1}{\Gamma 2}$$

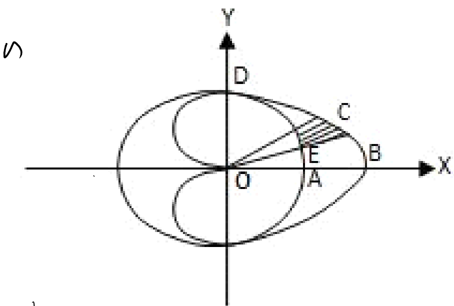
$$= \frac{a^2}{4} - \frac{a^2}{16} + \frac{a^2}{16}e^{-4} = \frac{3a^2}{16} + \frac{a^2 e^{-4}}{16}$$

4. Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$

Sol^n: $r = 2 \Rightarrow r^2 = 4 \Rightarrow x^2 + y^2 = 2^2$

This is a circle with centre at origin and radius 2

The two curves are as shown in figure.



The Region R outside the circle and inside the cardioide is ABCDEA and in polar coordinates

$dA = r\,dr\,d\theta$

On this strip, r varies from $r = 2$ to $r = 2(1+\cos\theta)$ and then θ varies from 0 to $\frac{\pi}{2}$ in the first quadrant

$$\therefore \iint_R \sin\theta \, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_2^{2(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \int_0^{\pi/2} [2(1+\cos\theta)^2 - 2] \sin\theta \, d\theta$$

$$= \int_0^{\pi/2} \left[2(1+\cos\theta)^2 - 2\right] \sin\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \left(1 + 2\cos\theta + \cos^2\theta - 1\right) \sin\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \left(2\sin\theta \cos\theta + \sin\theta \cos^2\theta\right) d\theta$$

$$= 4 \cdot \frac{1}{2} B\left(\frac{1+1}{2}, \frac{1+1}{2}\right) + 2 \cdot \frac{1}{2} B\left(\frac{1+1}{2}, \frac{2+1}{2}\right)$$

$$= 2B(1,1) + B\left(1, \frac{3}{2}\right)$$

$$= 2 + \frac{\Gamma 1 \, \Gamma 3/2}{\Gamma 5/2} = 2 + \frac{\frac{1}{2}\Gamma\frac{1}{2}}{\frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}$$

$$= 2 + \frac{2}{3} = \frac{8}{3}$$