CHANGE OF ORDER OF INTEGRATION:

To evaluate a double integral we integrate first the inner integral w.r.t. one variable (y or x depending upon the limits and the elementary strip) considering the other variable as constant and then integrate the outer integral with repect to the remaining variable.

However, if the limits are constants, as stated earlier, the order of integration is immaterial.

But if the limits are variable and the integrand $f(x, y)$ in the double integral is either difficult or even, sometimes, impossible to integrate in the given order than we reverse the order of integration and corresponding change is made in the limits of integration.

The new limits are obtained by geometrical considerations and therefore a clear sketch of the curve is to be drawn.

Sometimes in changing the order of integration we have to split up the region of integration and the new integral is expressed as the sum of a number of double integrals.

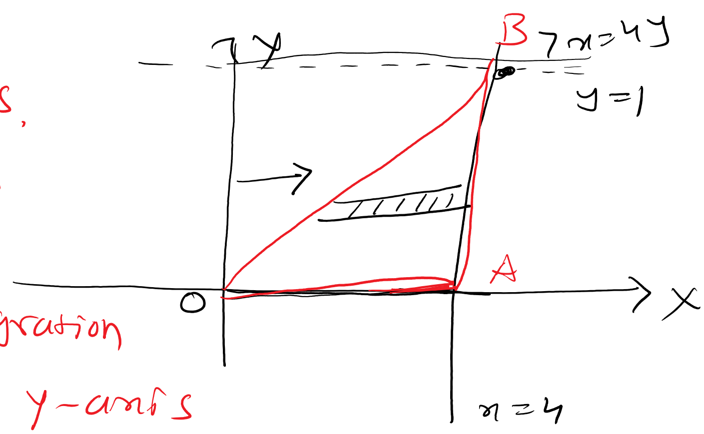
Change the order of following integrals and evaluate (if possible).

1. $\int_0^1 \int_{4y}^4 e^{x^2} dx\, dy$

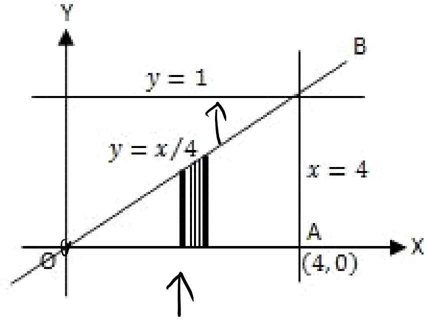
According to the given problem, the order of integration is first wrt x and then wrt y

The region of integration is $x = 4y$, a line through origin and $x = 4$, a line parallel to y axis $y = 0$; ie the x-axis and $y = 1$, a line parallel to x axis.

According to the given limits, the region of integration is OAB.



To change the order of integration consider a strip parallel to y-axis



on this vertical strip, y varies from $y=0$ to $y=\frac{x}{4}$ and then x varies from $x=0$ to $x=4$

$$\therefore I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx$$

$$= \int_0^4 e^{x^2} \left[y \right]_0^{x/4} dx = \int_0^4 \frac{x}{4} e^{x^2} dx$$

put $x^2 = t$ $\quad 2x\,dx = dt$

x	0	4
t	0	16

$$\therefore I = \frac{1}{4}\int_0^{16} e^t \, \frac{dt}{2} = \frac{1}{8}\int_0^{16} e^t dt = \frac{1}{8}\left(e^t\right)_0^{16}$$

$$I = \frac{1}{8}\left(e^{16} - 1\right)$$

2. $\int^1 \int^{\sqrt{1-x^2}} \quad e^y \quad dy\, dx$

2. $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx$

Soln:- The limits of y are $y=0$ and $y=\sqrt{1-x^2}$ and for x are $x=0$ and $x=1$

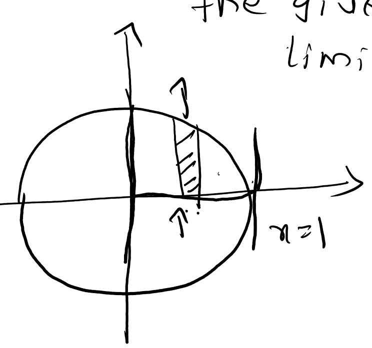
$y=0$:- x-axis

$y=\sqrt{1-x^2}$:- $y^2=1-x^2 \Rightarrow x^2+y^2=1$: circle

$x=0$:- y-axis

$x=1$: line parallel to y axis

According to the given limits



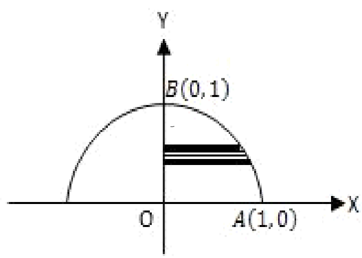
To change the order of integration we have to take a Horizontal strip.

The region of integration is OAB

In this region,

x varies from

$x=0$ to $x=\sqrt{1-y^2}$



$x^2+y^2=1$

$x^2=1-y^2$

$x=\sqrt{1-y^2}$

and then limit for y will be $y=0$ to 1

$$I=\int_0^1\int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dx\,dy$$

$$=\int_0^1 \frac{e^y}{e^y+1}\left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\right]dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\right] dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left(\sin^{-1}(1) - \sin^{-1}(0)\right) dy$$

$$= \int_0^1 \frac{e^y}{e^y+1} \cdot \frac{\pi}{2}\, dy = \frac{\pi}{2}\left[\log(e^y+1)\right]_0^1$$

$$= \frac{\pi}{2}\left[\log(e+1) - \log(1+1)\right]$$

$$I = \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

H.W **3.** $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$

4. $\int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\,dy$

Soln:- The limits for x are $2-\sqrt{4-y^2}$ and $2+\sqrt{4-y^2}$ and limits for y are 0 and 2.

$x = 2 - \sqrt{4-y^2}$

$(x-2) = -\sqrt{4-y^2}$

$(x-2)^2 = 4-y^2$

$(x-2)^2 + y^2 = 4$

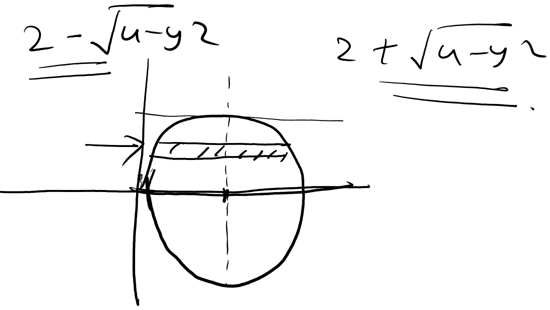
$x = 2 + \sqrt{4-y^2}$

$(x-2) = \sqrt{4-y^2}$

$(x-2)^2 = 4-y^2$

$(x-2)^2 + y^2 = 4$

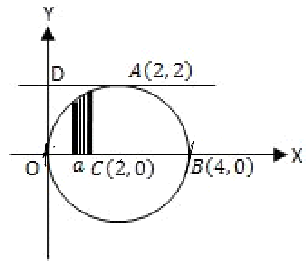
This is a circle with centre (2,0) and radius 2



To change the order of integration, we will consider a vertical strip.

On this strip, y varies from $y=0$ to

$$y=\sqrt{4-(x-2)^2}$$



$$(x-2)^2+y^2=4$$

$$y^2=4-(x-2)^2$$

$$y=\sqrt{4-(x-2)^2}$$

Then x varies from $x=0$ to $x=4$

$$I=\int_0^4\int_0^{\sqrt{4-(x-2)^2}}(1)\,dy\,dx$$

$$=\int_0^4\Big[y\Big]_0^{\sqrt{4-(x-2)^2}}dx$$

$$=\int_0^4\sqrt{4-(x-2)^2}\,dx$$

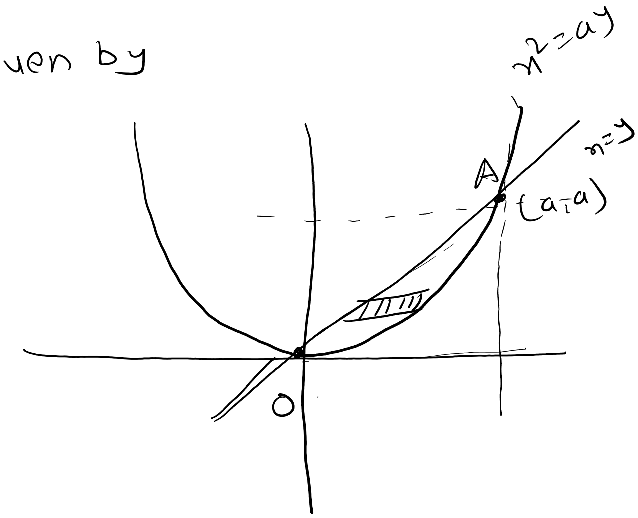
$$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$I=\left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2}+\frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right]_0^4$$

$I = 2\pi.$

5. $\int_0^a \int_y^{\sqrt{ay}} \frac{x}{x^2+y^2}\, dx\, dy$

The region of integration is given by



$x = y$: line through origin

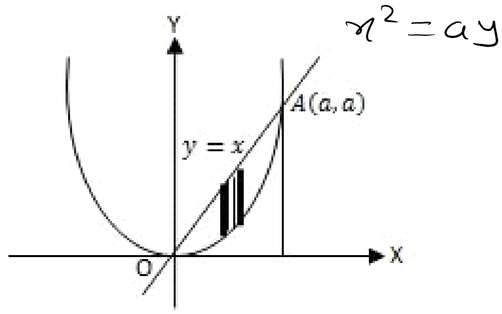
$x = \sqrt{ay} \Rightarrow x^2 = ay$: a parabola

Opening upwards

$y = 0$: x-axis

$y = a$: line parallel to x-axis

To change the order of integration, we consider a vertical strip in the region



On this strip,

y varies from

$y = \frac{x^2}{a}$ to $y = x$

Then x varies from

$x = 0$ to $x = a$.

$$I = \int_0^a \int_{x^2/a}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^a x \cdot \frac{1}{x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]_{x^2/a}^{x} dx$$

0

$$= \int_0^a \tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}\left(\frac{x^2/a}{x}\right) dx$$

$$= \int_0^a \frac{\pi}{4} - \tan^{-1}\left(\frac{x}{a}\right) dx$$

Integrating by parts

$$= \frac{\pi}{4}(x)_0^a - \left[x \cdot \tan^{-1}\left(\frac{x}{a}\right) - \int x \cdot \frac{1}{1+x^2/a^2} \cdot \frac{1}{a}\, dx\right]_0^a$$

$$= \frac{\pi}{4}(a) - \left[x \cdot \tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\int \frac{2x}{x^2+a^2}\, dx\right]_0^a$$

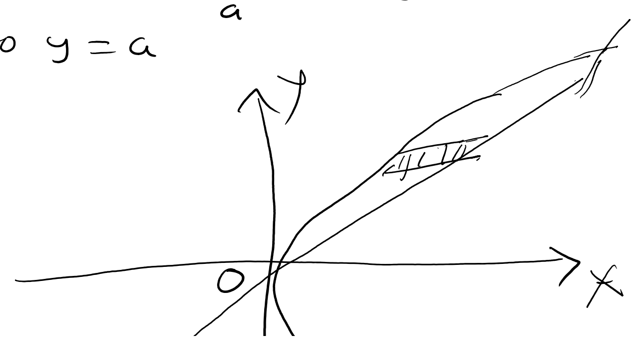
$$= \frac{\pi}{4}(a) - \left[x \cdot \tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\log\left(x^2+a^2\right)\right]_0^a$$

$$= \frac{\pi}{4}(a) - \left[a\tan^{-1}(1) - \frac{a}{2}\log\left(a^2+a^2\right) - 0 + \frac{a}{2}\log(a^2)\right]$$

$$I = \frac{a}{2}\log 2$$

6. $\int_0^a \int_{y^2/a}^{y} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dx\, dy$

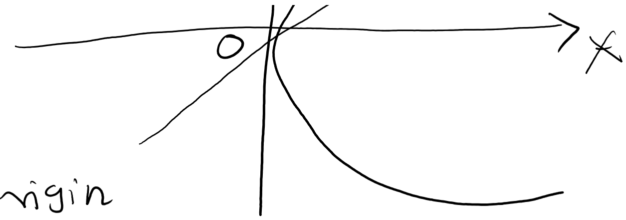
In the given region, x varies from $\frac{y^2}{a}$ to y and y varies from $y=0$ to $y=a$



$$x = \frac{y^2}{a} \Rightarrow y^2 = ax$$

This is a parabola

This is a parabola opening right side



$x=y$! a line passing thro' origin

$y=0$:- x-axis

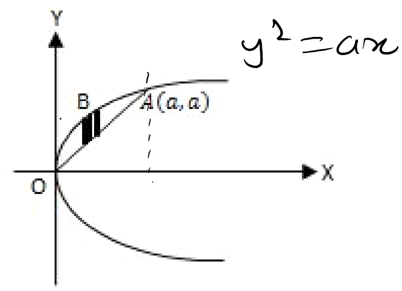
$y=a$! line parallel to x-axis

To change the order of integration, a vertical strip is to be considered in the region

In this strip,

y varies from $y=x$ to $\sqrt{ax}$

and then x ~~varies~~ varies from $x=0$ to $x=a$



$$\therefore I = \int_0^a \int_x^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\, dy\, dx$$

$$= \int_0^a \frac{1}{a-x} \left[\int_x^{\sqrt{ax}} \frac{y}{\sqrt{ax-y^2}}\, dy \right] dx$$

put $ax-y^2=t$ $\quad -2y\,dy = dt$

y	x	$\sqrt{ax}$
t	$ax-x^2$	0

$$= \int_0^a \frac{1}{a-x} \left[\int_{ax-x^2}^{0} \frac{-dt/2}{\sqrt{t}} \right] dx$$

$$0 \qquad ax-x^2$$

$$= \int_0^a \frac{1}{a-x} \left[\frac{1}{2}\int_0^{ax-x^2} t^{-\frac{1}{2}}\, dt\right] dx$$

$$= \int_0^a \frac{1}{a-x} \left[\frac{2\sqrt{t}}{2}\right]_0^{ax-x^2} dx$$

$$= \int_0^a \frac{1}{a-x} \left(\sqrt{ax-x^2}\right) dx = \int_0^a \frac{\sqrt{ax-x^2}}{a-x}\, dx$$

$$= \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\, dx \qquad \left(\text{or } \underline{\underline{x = at}}\right)$$

put $x = a\cos^2\theta$ $\quad \therefore dx = -2a\cos\theta\sin\theta\, d\theta$

x	0	a
θ	$\pi/2$	0

$$I = \int_{\pi/2}^{0} \frac{\sqrt{a}\cos\theta}{\sqrt{a}\sqrt{1-\cos^2\theta}} \left(-2a\cos\theta\sin\theta\, d\theta\right)$$

$$= 2a\int_0^{\pi/2} \cos^2\theta\, d\theta \qquad \left[= a\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta\right]$$

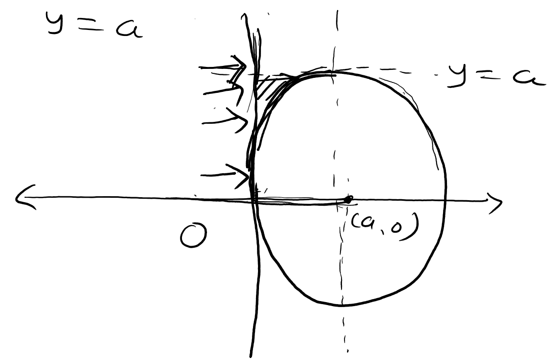
$$= 2a \cdot \frac{1}{2} B\left(\frac{2+1}{2}, \frac{0+1}{2}\right) = a\, B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\pi}{2} a$$

7. $\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2} dx$

In the given form, the integral is to be evaluated wrt x first, which is complicated.

we, therefore, change the order of integration.

The limits of x are $x=0$ to $x = a-\sqrt{a^2-y^2}$ and limits of y are $y=0$ to $y=a$



$x=0$: y-axis

$x = a \quad \sqrt{a^2-y^2} \quad a-\sqrt{a^2-y^2}$

$x-a = -\sqrt{a^2-y^2} \rightarrow$

$(x-a)^2 = a^2-y^2$.

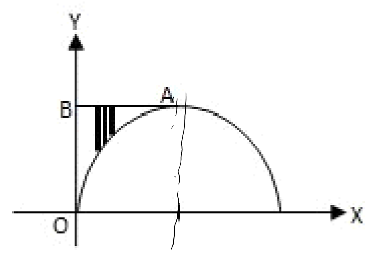
$(x-a)^2+y^2 = a^2 \rightarrow$ This is a circle with centre at $(a,0)$ and radius a.

The limits are from y-axis to left half of the circle $(x-a)^2+y^2=a^2$.

$\therefore$ If $A(a,a)$ and $B(0,a)$ then OAB is the region of integration.

Now to change the order of integration, we consider a strip parallel to y-axis

On this strip, y varies from the circle to line AB



$(x-a)^2+y^2 = a^2$

$y^2 = a^2-(x-a)^2$

$$y^2 = a^2 - (x-a)^2$$

$$y = \sqrt{a^2 - (x-a)^2} = \sqrt{2ax - x^2}$$

$\therefore$ y varies from $\sqrt{2ax - x^2}$ to $y = a$

and then x varies from $x = 0$ to $x = a$

$$I = \int_0^a \int_{\sqrt{2ax - x^2}}^a \frac{xy \log(x+a)}{(x-a)^2} \, dy \, dx$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} \left[\frac{y^2}{2} \right]_{\sqrt{2ax-x^2}}^a dx$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} \frac{1}{2} \left[a^2 - (2ax - x^2) \right] dx$$

$$= \frac{1}{2} \int_0^a \frac{x \log(x+a)}{(x-a)^2} (x-a)^2 \, dx = \frac{1}{2} \int_0^a x \log(x+a) \, dx$$

$$= \frac{1}{2} \left[\log(x+a) \cdot \frac{x^2}{2} - \int \frac{1}{x+a} \cdot \frac{x^2}{2} dx \right]_0^a$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \int \frac{x^2}{x+a} dx \right]_0^a$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \int \frac{(x^2 - a^2) + a^2}{x+a} dx \right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int (x-a)\,dx - \frac{1}{2}\int \frac{a^2}{x+a}\,dx \right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\left(\frac{x^2}{2} - ax\right) - \frac{1}{2}a^2\log(x+a) \right]_0^a$$

$$I = \frac{a^2}{8}\left[1 + 2\log a\right]$$

H.W **8.** $\int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}}\,dy\,dx$

H.W **9.** $\int_0^{\pi} \int_0^x \frac{\sin y}{\sqrt{(\pi-x)(x-y)}}\,dy\,dx$

H.W **10.** $\int_0^a \int_0^y \frac{x}{\sqrt{(a^2-x^2)(a-y)(y-x)}}\,dx\,dy$

H.W **11.** $\int_0^a \int_0^x \frac{\sin y}{\sqrt{(a-x)(x-y)(4-5\cos y)^2}}\,dy\,dx$

H.W **12.** $\int_0^2 \int_{\sqrt{2x}}^2 \frac{y^2}{\sqrt{y^4-4x^2}}\,dy\,dx$

H.W **13.** $\int_0^{1/2} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy$

14. $\int_{x=0}^{1} dx \int_{y=1}^{\infty} e^{-y} y^x \log y\,dy$

Soln:- since all the limits of integration are constants, the order can be changed without taking the help of a diagram

$$I = \int_{y=1}^{\infty} e^{-y}\log y\,dy \left[\int_{x=0}^{1} y^x\,dx\right]$$

$$\int^{\infty} e^{-y}\log y\,dy \left(\frac{y^x}{\log y}\right)^1$$

$$= \int_{y=1}^{\infty} e^{-y} \log y \, dy \left[\frac{y^x}{\log y} \right]_{x=0}^{1}$$

$$= \int_{y=1}^{\infty} e^{-y} \log y \left[\frac{y}{\log y} - \frac{1}{\log y} \right] dy$$

$$= \int_{1}^{\infty} e^{-y} (y-1) \, dy$$

$$= \left(-y e^{-y} \right)_{1}^{\infty} = \frac{1}{e}$$

H.W

15. Change the order of integration and evaluate the integral $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dy\, dx$

16. $\int_0^5 \int_{2-x}^{2+x} dy\, dx$ change the order.

The region of integration is bounded by

$y = 2 - x$ to $y = 2 + x$

both are lines $x + y = 2$ and $y - x = 2$

and then x limits are $x = 0$ to $x = 5$

$x + y = 2$

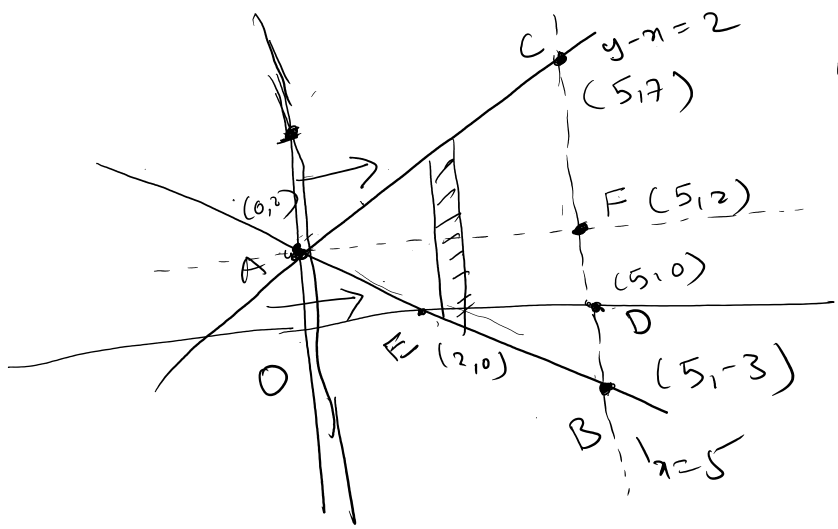
$x = 0 \quad y = 2$

$x = 2 \quad y = 0$

$y - x = 2$

$x = 0 \quad y = 2$

$x = -2 \quad y = 0$



AEDC

ABC

BDE

$x + y = 2$

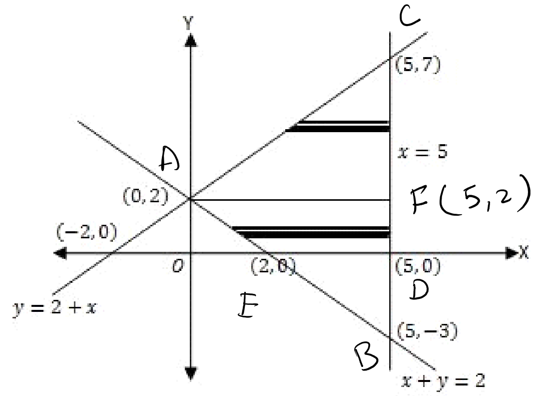
The region of integration is ABC

When the order is changed, the region splits into two parts

In the region AFC,

x varies from $y-2$ to 5

y varies from 2 to 7



In the region ABF,

x varies from $2-y$ to 5

y varies from -3 to 2

$$I = \int_{-3}^{2}\int_{2-y}^{5} dx\, dy + \int_{2}^{7}\int_{y-2}^{5} dx\, dy$$

Evaluation is H.W   Answer is $I = 25$

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17. $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$

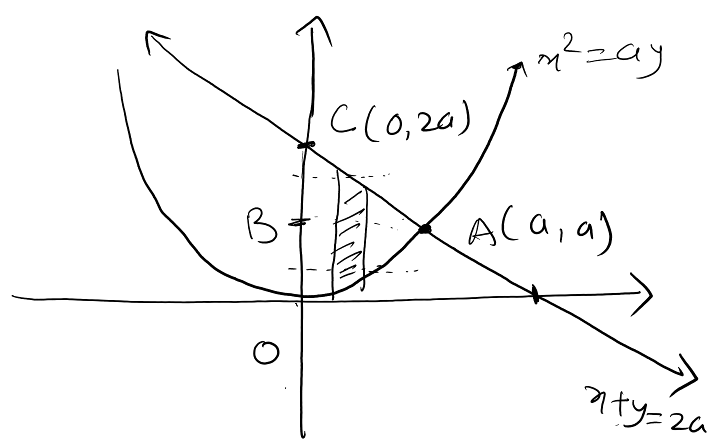
The limits for y are $\frac{x^2}{a}$ and $2a-x$ and those for x are 0 to a.

$y = \frac{x^2}{a} \Rightarrow x^2 = ay$: this a parabola opening upwards

$y = 2a - x \Rightarrow x + y = 2a$: this a line

$x=0 \quad y = 2a$

$x=2a \quad y=0$



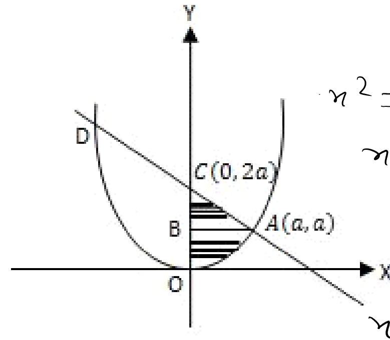
The point A, point of intersection of $x^2 = ay$ and $x+y=2a$.

$A \equiv (a,a)$

The region of integration is OAC

Now, change the order of integration, if we consider a strip parallel to the x-axis

The region has to be divided into two parts OAB and BAC.



$x^2 = ay$

$x = \sqrt{ay}$

$x+y = 2a$

In the region, OAB

x varies from 0 to $\sqrt{ay}$

and y varies from 0 to a.

In the region BAC

x varies from 0 to $2a-y$

y varies from a to $2a$

$$I = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

Evaluation is H.W. Final answer $= \frac{3}{8} a^4$

H.W

18. $\int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy$

H.W

19. $\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dxdy$

H.W

20. $\int\int_R x^2 dx\, dy$ where R is the region in the first quadrant bounded by $xy = a^2$, $x = 2a, y = 0$ and $y = x$

Express as a single integral and then evaluate.

21. $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^3 dy \int_{-1}^1 dx$

let $I = I_1 + I_2$

For I_1, the limits are $x = -\sqrt{y}$ and $x = \sqrt{y}$

ie $x^2 = y$: a parabola with vertex at origin and opening upwards

The limits for y are 0 to 1

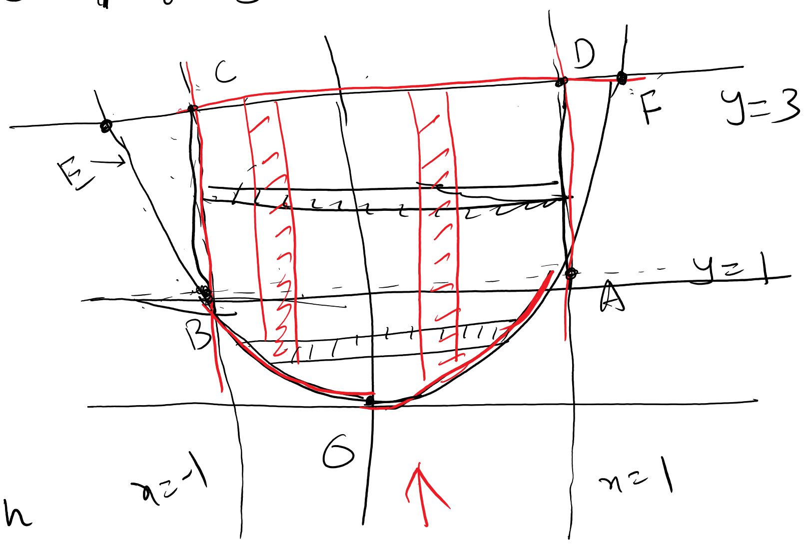
For I_2, x limits are -1 to 1

y limits are 1 to 3

For I_1, OAB is the region of Integration

For I_2, ABCD is the region of integration

we have to Combine both the regions ie the region OADCBO



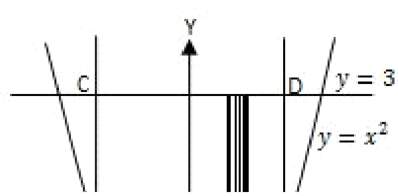
Now consider a strip parallel to y-axis extending from the parabola to the line CD.

On this strip y-varies from $y = x^2$ to $y = 3$

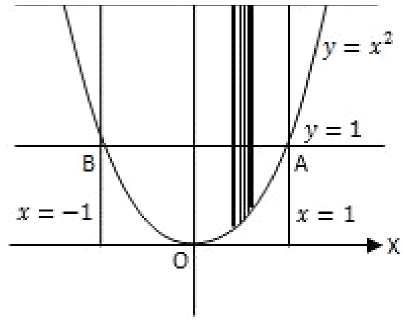
To sweep the whole area the strip has to move from $x = -1$ to $x = 1$

$$\therefore I = \int_{-1}^{1} \int_{x^2}^{3} dy\, dx$$

1



$$-1 \quad x^2$$



$$= \int_{-1}^{1} \left(y\right)_{x^2}^{3} dx$$

$$= \int_{-1}^{1} \left(3 - x^2\right) dx$$

$$= \left(3x - \frac{x^3}{3}\right)_{-1}^{1}$$

$$= \frac{16}{3}$$