EVALUATION OVER GIVEN LIMITS

Monday, May 3, 2021 12:00 PM

1) $\int_0^1 \int_0^y xye^{-x^2} dx\, dy$

Here we first evaluate inner integral wrt x

$$I = \int_0^1 \left[\int_0^y x e^{-x^2} dx \right] y\, dy$$

put $-x^2 = t$ $\quad -2x\,dx = dt$

$x = 0 \quad t = 0 \qquad x = y \quad t = -y^2$

$$= \int_0^1 \left[\int_0^{-y^2} e^t \left(-\frac{dt}{2}\right) \right] y\, dy$$

$$= \int_0^1 \left[-\frac{1}{2} \left(e^t\right)_0^{-y^2} \cdot y\, dy \right] = \int_0^1 -\frac{1}{2}\left(e^{-y^2} - 1\right) y\, dy$$

$$= -\frac{1}{2} \int_0^1 y e^{-y^2} dy + \frac{1}{2} \int_0^1 y\, dy$$

put $-y^2 = t$ $\quad -2y\, dy = dt$

$y = 0 \quad t = 0 \quad | \quad y = 1 \quad t = -1$

$$I = \int_0^{-1} -\frac{1}{2} e^t \left(-\frac{dt}{2}\right) + \frac{1}{2}\left(\frac{y^2}{2}\right)_0^1$$

$$= \frac{1}{4}\left(e^t\right)_0^{-1} + \frac{1}{4} = \frac{1}{4}\left(e^{-1} - 1\right) + \frac{1}{4} = \frac{1}{4e}$$

2) $\int_0^1 \int_0^{x^2} e^{(y/x)} dy\, dx$

Sol^n:- Let $I = \int_0^1 \int_0^{x^2} e^{(y/x)}\, dy\, dx$

Here we integrate wrt y first

$$I = \int_0^1 \left[\int_0^{x^2} e^{(y/x)} dy \right] dx$$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} dx$$

$$= \int_0^1 x\left(e^x - 1\right) dx = \int_0^1 x e^x\, dx - \int_0^1 x\, dx$$

integrating by parts

$$= \left[x \int e^x dx - \int (1)\, e^x\, dx\right]_0^1 - \left(\frac{x^2}{2}\right)_0^1$$

$$= \left(x e^x - e^x\right)_0^1 - \frac{1}{2} = (e - e) - (0 - 1) - \frac{1}{2}$$

$$\therefore I = \frac{1}{2}$$

3) $\int_0^{\infty} \int_0^{\infty} e^{-x^2(1+y^2)} x\, dx\, dy$

$\int^{\infty} \int^{\infty} e^{-x^2(1+y^2)}$

3) $\int_0 \int_0 e^{-x^2(1+y^2)} x\,dx\,dy$

$$I = \int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} \cdot \underline{x\,dx}\,dy$$

integrating wrt x first

put $x^2(1+y^2) = t$

$$2x(1+y^2)\,dx = dt$$

when $x=0$, $t=0$; when $x=\infty$, $t=\infty$

$$I = \int_0^\infty \left[\int_0^\infty e^{-t} \frac{dt}{2(1+y^2)}\right] dy$$

$$= \int_0^\infty \left[\int_0^\infty e^{-t}\,dt\right] \frac{1}{2(1+y^2)}\,dy$$

$$= \int_0^\infty \left(-e^{-t}\right)_0^\infty \cdot \frac{1}{2(1+y^2)}\,dy$$

$$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$= \int_0^\infty -\left(e^{-\infty} - e^0\right) \cdot \frac{1}{2(1+y^2)}\,dy = \int_0^\infty \frac{1}{2(1+y^2)}\,dy$$

$$= \frac{1}{2}\left(\tan^{-1} y\right)_0^\infty = \frac{1}{2}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

4) $\int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$

Soln: integrate wrt y first

$$I = \int_0^1 \left[\int_{x^2}^{x} (x^2 y + x y^2)\,dy \right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left(\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1 = \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$

5) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$

Soln: integrating wrt y first

$$I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2}\,dy \right] dx$$

$$\left[\int \frac{1}{a^2+y^2}\,dy = \frac{1}{a} \tan^{-1}\left(\frac{y}{a}\right) \right.$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)_0^{\sqrt{1+x^2}} dx$$

$$= \int^1$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2}\right)\right]_0^1 = \frac{\pi}{4} \log(1+\sqrt{2})$$

H.W.

6) $\int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{1}{\sqrt{1-x^2-y^2}}\, \cancel{dy\, dx}$ Ans: $\frac{\pi}{4}$

Hint: integrate wrt x first

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7) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

Soln: $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\, dx\, dy$

Integrating wrt x first

$$\left[\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2-y^2)-x^2} + \left(\frac{a^2-y^2}{2}\right)\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int^a \left[0 + \left(\frac{a^2-y^2}{2}\right)\sin^{-1}(1) - 0 - \left(\frac{a^2-y^2}{2}\right)\sin^{-1}(0)\right] dy$$

$$= \int_0^a \left[0 + \left(\frac{a^2-y^2}{2}\right)\sin^{-1}(1) - 0 - \left(\frac{a^2-y^2}{2}\right)\sin^{-1}(0)\right] dy$$

$$= \int_0^a \left(\frac{a^2-y^2}{2}\right)\cdot\frac{\pi}{2}\, dy = \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a$$

$$I = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi a^3}{6}$$

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8) $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\, dx\, dy$

let $I = \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\, dx\, dy$

Integrating wrt x first

$$I = \int_1^2 2y^2 \left(\frac{x^3}{3}\right)_{-\sqrt{2-y}}^{\sqrt{2-y}} dy$$

$$= \int_1^2 \frac{2y^2}{3}\left[(2-y)^{3/2} - \left(-(2-y)\right)^{3/2}\right] dy$$

$$= \frac{4}{3}\int_1^2 y^2 (2-y)^{3/2}\, dy$$

let $2-y = t \Rightarrow -dy = dt$

when $y=1$, $t=1$ and when $y=2$, $t=0$

$$I = \frac{4}{3}\int_1^0 (2-t)^2 t^{3/2} (-dt) = \frac{4}{3}\int_0^1 t^{3/2}(4-4t+t^2)\,dt$$

$$= \frac{4}{3}\int_0^1 \left(4t^{3/2} - 4t^{5/2} + t^{7/2}\right)dt = \frac{856}{945}$$

9) $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

Let $\displaystyle I = \int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$

Integrating wrt r first

put $1+r^2=t$ $\quad 2r\,dr = dt$

when $r=0$, $t=1$ $\quad$ when $r=\sqrt{\cos 2\theta}$, $t=1+\cos 2\theta$

$$I = \int_0^{\pi/4}\int_1^{1+\cos 2\theta} \frac{1}{t^2}\frac{dt}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(-\frac{1}{t}\right)_1^{1+\cos 2\theta} d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/4}\left(\frac{1}{1+\cos 2\theta} - 1\right) d\theta$$

$$= \frac{-1}{2}\int_0^1 \left(\frac{1}{1+\cos 2\theta} - 1\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(1 - \frac{1}{2}\sec^2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{1}{2}\tan\theta\right)_0^{\pi/4} = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\tan\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{8} - \frac{1}{4}$$

10) $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr\, d\theta$

$$I = \int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr\, d\theta$$

Integrating wrt r first

put $a^2 - r^2 = t$

$-2r\,dr = dt$

when $r=0$ $t=a^2$ | when $r=a\cos\theta$, $t=a^2(1-\cos^2\theta)$
$= a^2\sin^2\theta$

$$I = \int_0^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} \sqrt{t}\left(-\frac{dt}{2}\right) d\theta$$

$\pi/2$ $a^2\sin^2\theta$

$$= -\frac{1}{2}\int_0^{\pi/2} \left(\frac{t^{3/2}}{3/2}\right)_{a^2}^{a^2\sin^2\theta} d\theta$$

$$= \frac{-1}{3}\int_0^{\pi/2} (a^3\sin^3\theta - a^3)\, d\theta$$

$$= \frac{a^3}{3}\left[\int_0^{\pi/2}(1-\sin^3\theta)\, d\theta\right]$$

$$= \frac{a^3}{2}\left[\int_0^{\pi/2} d\theta - \int_0^{\pi/2}\sin^3\theta\, d\theta\right]$$

$$\left[\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)\right.$$

$$= \frac{a^3}{2}\left[(\theta)_0^{\pi/2} - \frac{1}{2}B\left(\frac{3+1}{2}, \frac{1}{2}\right)\right]$$

$$= \frac{a^3}{2}\left[\frac{\pi}{2} - \frac{1}{2}\frac{\Gamma 2\,\Gamma 1/2}{\Gamma 5/2}\right] = \frac{a^3}{2}\left[\frac{\pi}{2} - \frac{1}{2}\frac{\Gamma 1/2}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}\right]$$

$$I = \frac{a^3}{2}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

11) $\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta\, dr d\theta$

$$I = \int_0^{\pi/2}\int^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$$

$$\int_0 \int_0 r^2 \cos\theta \, dr \, d\theta$$

we integrate wrt r first

$$= \int_0^{\pi/2} \cos\theta \left(\frac{r^3}{3}\right)_0^{1-\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos\theta \, (1-\sin\theta)^3 \, d\theta$$

put $1-\sin\theta = t$

$-\cos\theta \, d\theta = dt$

$\theta = 0$, $t = 1$ | when $\theta = \pi/2$, $t = 0$

$$I = \frac{1}{3}\int_1^0 t^3 (-dt) = \frac{1}{3}\int_0^1 t^3 \, dt$$

$$= \frac{1}{3}\left(\frac{t^4}{4}\right)_0^1$$

$$I = \frac{1}{12}$$