CONCEPT

Monday, May 3, 2021 11:30 AM

EVALUATION OF DOUBLE INTEGRALS:

To evaluate a double integral we integrate first the inner integral w.r.t one variable (y or x depending upon the limits and the elementary strip) considering the other variable as constant and then integrate the outer integral with respect to the remaining variable.

$$\int_0^1\int_0^y (x^2+y^2)\,dx\,dy \qquad \int_0^1\left[\frac{x^3}{3}+y^2x\right]_0^y dy$$

$$= \int_0^1\left(\frac{y^3}{3}+y^3\right)dy = \left(\frac{y^4}{12}+\frac{y^4}{4}\right)_0^1 = \frac{1}{12}+\frac{1}{4}=\frac{1}{3}$$

However, if the limits are constants, the order of integration is immaterial.

Note:

(1) Always integrate from inside outwards, i.e., evaluate inner integral first and outer integral last.

(2) If limits of inner integral are functions of x i.e., y in terms of x then integrate it w.r.t y and if functions of y i.e., x in terms of y then integrate it w.r.t x

$$\int_0^1\int_0^x x^2y\,dx\,dy \rightarrow \text{integrate wrt } y \text{ first}$$

$$\int_0^1\int_0^{y^2} x^2\,dx\,dy \rightarrow \text{integrate wrt } x \text{ first}$$

(3) Limits of outer most integral are always constant numbers.

(4) If limits of both the integrals are constants and if variables can be separated then the given double integral can be expressed as a product of two integrals as follows:

$$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_a^b\int_c^d g(x).\emptyset(y)\,dx\,dy = \left[\int_a^b g(x)dx\right]\left[\int_c^d \emptyset(y)dy\right]$$

$$\int_{x=0}^1\int_{y=0}^2 x^2y\,dx\,dy = \left[\int_0^1 x^2\,dx\right]\left[\int_0^2 y\,dy\right] = \left(\frac{x^3}{3}\right)_0^1\left(\frac{y^2}{2}\right)_0^2$$

$$= \frac{1}{3}\times 2 = \frac{2}{3}$$

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$$\int_{x=0}^{1}\int_{y=0}^{2} x^2 y \, dx\, dy = \int_0^1 x^2 \cdot \left(\frac{y^2}{2}\right)_0^2 dx = \int_0^1 2x^2\, dx = \left(\frac{2x^3}{3}\right)_0^1 = \frac{2}{3}$$

(5) If limits of both the integrals are constants and if variables can be or cannot be separated then the order of integration can be reversed as follows:

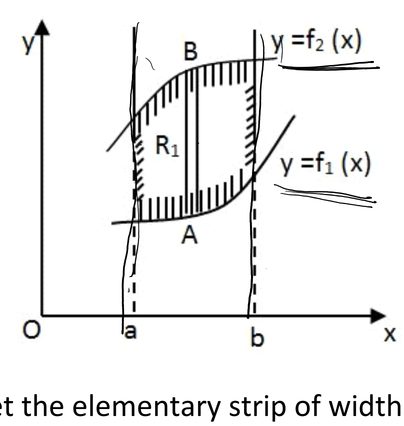
$$\int_a^b \int_c^d f(x,y) dx\, dy = \int_c^d \int_a^b f(x,y) dy\, dx$$

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2 y \, dx\, dy = \frac{2}{3}$$ (evaluated wrt y first and then wrt x)

$$\int_0^2\int_0^1 x^2 y \, dx\, dy = \int_0^2 \left(\frac{x^3}{3}\right)_0^1 y\, dy = \int_0^2 \frac{y}{3}\, dy = \left(\frac{y^2}{6}\right)_0^2 = \frac{2}{3}$$

GEOMETRICAL REPRESENTATION:

1) Let the region of integration be R_1 (the shaded area) which is the area bounded by the curves $y = f_1(x), y = f_2(x)$ and the lines $x = a, x = b$.



$$\int_a^b \int_{f_1(x)}^{f_2(x)} \phi(x,y)\, dy\, dx$$

Let the elementary strip of width δx, parallel to y – axis, (such as the vertical strip AB) and extends from $y = f_1(x)$ to $y = f_2(x)$, which are y – limits.

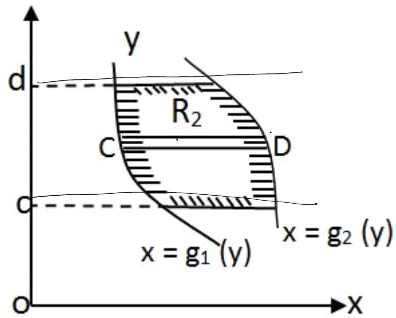
To cover the entire region of integration the strip goes on sliding from $x = a \text{ to } x = b$, (i.e. from left to right) which are x – limits.

Then the integration of the function $f(x, y)$ taken over the region R_1 is given by

$$I = \int_{x=a}^{x=b} dx \int_{y=f_1(x)}^{y=f_2(x)} f(x,y) dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) dy\ dx$$

$$I = \int_{x=a}^{x=b} dx \int_{y=f_1(x)}^{y=f_2(x)} f(x,y)dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dy\ dx$$

Similarly, let the region of integration be R_2 (the shaded area), which is the area bounded by the curves $x = g_1(y), x = g_2(y)$ and the lines $y = c, y = d$.



$$\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\ dx\ dy$$

Let the elementary strip of width δy, parallel to x – axis (such as the horizontal strip CD) and extends from $x = g_1(y)$ to $x = g_2(y)$, which are x – limits.

To cover the entire region of integration such strip goes on sliding from $y = c \ to \ y = d$ (i.e., from down to up), which are y – limits.

Then the integration of the function $f(x, y)$ taken over the region R_2 is given by

$$I = \int_{y=c}^{y=d} dy \int_{x=g_1(y)}^{x=g_2(y)} f(x,y)dx = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dx\ dy$$

Note:

(1) Any double integral can be performed in both the ways either by taking the elementary strips parallel to y – axis or x – axis i.e., the order of integration can be changed. But the values of integral in both the cases is same.

(2) x – limits are to be taken from left to right (i.e. left value of x is for lower limit and right value of x is for upper limit). And y – limits are to be taken from down to up.

(3) In various problems on double integral while evaluation of integration the transformation of cartesian system into polar system is found convenient by putting $x = r\cos\theta$ and $y = r\sin\theta$, specially when the region of integration is a circle, a lemniscate, etc