RECTIFICATION - POLAR CURVES

Tuesday, April 27, 2021 11:30 AM

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

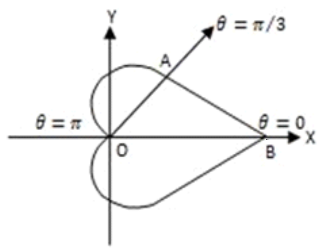
LENGTH OF THE ARC OF A CURVE GIVEN IN POLAR FORM

- **(i)** Length of the arc of a curve given by $r = f(\theta)$ is $S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

(ii) Length of the arc of a curve given by $\theta = f(r)$ is $S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$

1) Find the length of the perimeter $r = a(1+\cos\theta)$. Prove also that the upper half of cardiode is bisected by the line $\theta = \pi/3$.

$$r = a(1+\cos\theta)$$



Total perimeter = 2 arc OB

$$= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = a(1+\cos\theta) \rightarrow \frac{dr}{d\theta} = a(-\sin\theta)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\left[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\right] = 2a^2(1+\cos\theta)$$

$$= 4a^2\cos^2\frac{\theta}{2}$$

$$\therefore \text{perimeter} = 2\int_0^{\pi} \sqrt{4a^2\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 4a \int_0^{\pi} \cos\frac{\theta}{2}\, d\theta$$

$$= 4a \left[2 \sin\frac{\theta}{2} \right]_0^{\pi}$$

$$= 8a \left[\sin\frac{\pi}{2} - \sin 0 \right]$$

Total perimeter $= 8a$

For the second part, the arc where the line $\theta = \frac{\pi}{3}$ divides the cardioide is given by

$$\text{Arc } AB = \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/3} 2a \cos\frac{\theta}{2}\, d\theta \quad \left(\text{using the } 1^{st} \text{ part}\right)$$

$$= 2a \left[2 \sin\frac{\theta}{2} \right]_0^{\pi/3} = 4a \left[\sin\frac{\pi}{6} - \sin 0 \right]$$

$\therefore$ Arc $AB = 4a \left[\frac{1}{2} \right] = 2a$

Also length of Arc $OB = \frac{1}{2}$ perimeter $= 4a$

$\therefore$ Arc $AB = \frac{1}{2}$ Arc OB

$\therefore$ The line $\theta = \frac{\pi}{3}$ bisects the upper half of the cardioide.

2) Find the length of the arc of the curve $r = a \sin^2\left(\frac{\theta}{2}\right)$ from $\theta = 0$ to any point $P(\theta)$

Soln:- The required length is given by

$$S = \int_0^{\theta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = a\sin^2\frac{\theta}{2} \rightarrow \frac{dr}{d\theta} = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\frac{1}{2}$$

$$= a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Now $r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^4\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$

$$= a^2\sin^2\frac{\theta}{2}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)$$

$$= a^2\sin^2\frac{\theta}{2}$$

The required length $= \int_0^{\theta} a\sin\frac{\theta}{2}\, d\theta$

$$= a\left[-2\cos\frac{\theta}{2}\right]_0^{\theta}$$

$$= 2a\left[1 - \cos\frac{\theta}{2}\right]$$

The required length. $= 4a\sin^2\frac{\theta}{4}$

3) Find the length of the cardioide $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$

$$r = a\cos\theta$$

$$r^2 = a\, r\cos\theta$$

$$x^2 + y^2 = ax$$

$$x^2 - ax + y^2 = 0$$

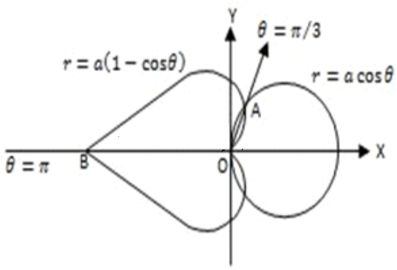
$$x^2 - ax + y^2 = 0$$

$$\left(x^2 - ax + \frac{a^2}{4}\right) + y^2 = \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

circle with centre at $\left(\frac{a}{2}, 0\right)$ radius $\frac{a}{2}$

The circle and the cardioide are shown in figure.



They intersect where

$$a(1 - \cos\theta) = a\cos\theta$$

$$1 - \cos\theta = \cos\theta$$

$$\Rightarrow 2\cos\theta = 1$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

The length of the cardioide outside the circle is 2 Arc AB where for A, $\theta = \pi/3$ and for B $\theta = \pi$.

$$\therefore \text{ required length} = 2\int_{\pi/3}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Now $r = a(1 - \cos\theta)$

$$\frac{dr}{d\theta} = a(\sin\theta)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$

$$= 2a^2(1 - \cos\theta) = 4a^2\sin^2\frac{\theta}{2}$$

$\therefore$ The required length $= 2\int_{\pi/3}^{\pi} 2a\sin\frac{\theta}{2}\,d\theta$

$$= 4a\left[-2\cos\frac{\theta}{2}\right]_{\pi/3}^{\pi}$$

$$= -8a\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right]$$

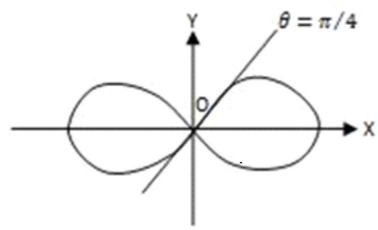
$$= -8a\left[0 - \frac{\sqrt{3}}{2}\right]$$

length of cardioide outside the circle $= 4\sqrt{3}\,a$.

4) Find the length of the upper arc of one loop of Lemniscate $r^2 = a^2\cos 2\theta$

Sol:- The curve is shown in the figure

It is clear that for upper half of one loop. θ varies from 0 to $\pi/4$



$$\therefore S = \int_0^{\pi/4}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Now $r = a\sqrt{\cos 2\theta}$

$$\therefore \frac{dr}{d\theta} = a \cdot \frac{1(-2\sin 2\theta)}{2\sqrt{\cos 2\theta}}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}$$

$$= a^2\left[\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}\right] = \frac{a^2}{\cos 2\theta}$$

$$\therefore S = \int_0^{\pi/4} \frac{a}{\sqrt{\cos 2\theta}}\, d\theta$$

put $2\theta = t \qquad d\theta = \frac{dt}{2}$

$\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{4} \quad t = \frac{\pi}{2}$

$$S = \int_0^{\pi/2} \frac{a}{\sqrt{\cos t}} \cdot \frac{dt}{2} = \frac{a}{2}\int_0^{\pi/2} \cos^{-1/2} t \, \sin^0 t \, dt$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$S = \frac{a}{2}\cdot\frac{1}{2} B\left(\frac{0+1}{2}, \frac{-1/2+1}{2}\right) = \frac{a}{4} B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$S = \frac{a}{4}\cdot\frac{\Gamma(1/2)\,\Gamma(1/4)}{\Gamma(3/4)} = \frac{a\sqrt{\pi}}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

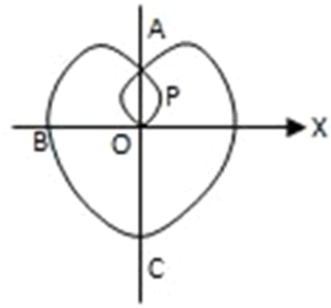
but $\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$

$$S = \frac{a\sqrt{\pi}}{\ldots} \quad \Gamma(1/4) \cdot \Gamma(1/4) \qquad a\,(\Gamma(1/4))^2$$

$$S = \frac{a\sqrt{\pi}}{4} \cdot \Gamma\tfrac{1}{4} \cdot \frac{\Gamma\tfrac{1}{4}}{\pi\sqrt{2}} = \frac{a\left(\Gamma\tfrac{1}{4}\right)^2}{4\sqrt{2}\sqrt{\pi}}$$

5) Find the total length of the curve $r = a\,sin^3(\theta/3)$

The curve is shown in figure



For half the arc OPABC,

θ varies from 0 to $\frac{3\pi}{2}$

Since, $r = a\sin^3\left(\frac{\theta}{3}\right)$

$$\frac{dr}{d\theta} = 3a\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\cdot\frac{1}{3}$$

$$= a\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^6\left(\frac{\theta}{3}\right) + a^2\sin^4\left(\frac{\theta}{3}\right)\cos^2\frac{\theta}{3}$$

$$= a^2\sin^4\left(\frac{\theta}{3}\right)\left[\sin^2\left(\frac{\theta}{3}\right)+\cos^2\left(\frac{\theta}{3}\right)\right]$$

$$= a^2\sin^4\left(\frac{\theta}{3}\right)$$

The required length $= 2\int_0^{3\pi/2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$= 2\int^{3\pi/2} a\sin^2\left(\frac{\theta}{3}\right)d\theta$$

$$= 2\int_0 a \sin^2\left(\frac{\theta}{3}\right) d\theta$$

$$= \int_0^{3\pi/2} a\left(1 - \cos\frac{2\theta}{3}\right) d\theta \qquad \left(2\sin^2\theta = 1 - \cos 2\theta\right)$$

$$= a\left[\theta - \sin\left(\frac{2\theta}{3}\right)\cdot\frac{3}{2}\right]_0^{3\pi/2}$$

$$= a\left[\frac{3\pi}{2} - \frac{3}{2}\sin(\pi) - 0 + \frac{3}{2}\sin 0\right]$$

The total length of given curve $= \frac{3}{2}\pi a$

6) Find the length of the arc of the parabola $r = \frac{6}{1+\cos\theta}$ from $\theta = 0$ to $\theta = \pi/2$

Soln:- since $r = \frac{6}{1+\cos\theta} = \frac{6}{2\cos^2\frac{\theta}{2}} = 3\sec^2\left(\frac{\theta}{2}\right)$

$$\frac{dr}{d\theta} = 3\cdot 2\sec\left(\frac{\theta}{2}\right)\cdot\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}$$

$$\frac{dr}{d\theta} = 3\sec^2\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 9\sec^4\left(\frac{\theta}{2}\right) + 9\sec^4\left(\frac{\theta}{2}\right) \ldots$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = 9\sec^4\left(\frac{\theta}{2}\right) + 9\sec^4\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\theta}{2}\right)$$

$$= 9\sec^4\left(\frac{\theta}{2}\right)\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)$$

$$r^2+\left(\frac{dr}{d\theta}\right)^2 = 9\sec^6\left(\frac{\theta}{2}\right)$$

The required length $= s = \int_0^{\pi/2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{\pi/2} 3\sec^3\left(\frac{\theta}{2}\right) d\theta$$

$\tan\left(\frac{\theta}{2}\right) = t$ | $\theta = 0 \quad t = 0$

$\sec^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}\, d\theta = dt$ | $\theta = \pi/2 \quad t = 1$

$$S = \int_0^1 3\cdot 2\, dt\, \sqrt{1+t^2}$$

$$= 6\int_0^1 \sqrt{1+t^2}\, dt$$

$$\left[\int \sqrt{1+x^2}\, dx = \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log\left(x+\sqrt{x^2+1}\right)\right]$$

$$S = 6\left[\frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log\left(t+\sqrt{t^2+1}\right)\right]_0^1$$

$$= 6\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1+\sqrt{2}) - 0 - \frac{1}{2}\log(1)\right]$$

$$S = 3\left[\sqrt{2}+\log(\sqrt{2}+1)\right]$$

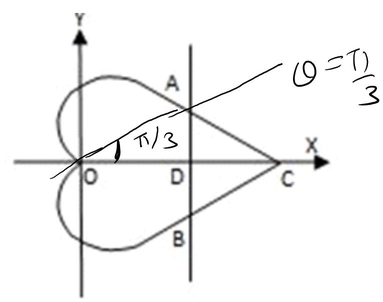
7. Show that the length of the arc of that part of cardioide $r = a(1+\cos\theta)$ which lies on the side of the line $4r = 3\,a sec\theta$ away from the pole is $4a$

OR Show that the perimeter of cardioid $r = a(1+\cos\theta)$ is bisected by the line $4\,r = 3a\sec\theta$

The cardioide is as shown in the figure

Now $4r = 3a\sec\theta$

$4r\cos\theta = 3a$



but $x = r\cos\theta$

$\therefore 4x = 3a \Rightarrow x = \frac{3a}{4}$

This is a line parallel to y-axis passing through $\left(\frac{3a}{4}, 0\right)$

Now, At the point of intersection A, solving $r = a(1+\cos\theta)$ and $4r = 3a\sec\theta$

$$r = \frac{3a\sec\theta}{4}$$

$$a(1+\cos\theta) = \frac{3a\sec\theta}{}$$

$$4$$

$$4a(1+\cos\theta) = 3a\sec\theta$$

$$4a(1+\cos\theta)\cos\theta = 3a$$

$$4\cos\theta + 4\cos^2\theta = 3$$

$$4\cos^2\theta + 4\cos\theta - 3 = 0$$

$$(2\cos\theta + 3)(2\cos\theta - 1) = 0$$

$2\cos\theta + 3 = 0$ or $2\cos\theta - 1 = 0$

$\cos\theta = -\frac{3}{2}$ which is not possible

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

length of arc ACB = 2 length of arc AC

$$= 2\int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$r = a(1+\cos\theta)$$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta$$

$$= 2a^2(1+\cos\theta)$$

$$= 4a^2\cos^2\frac{\theta}{2}$$

$$\therefore \text{arc } ACB = 2\int^{\pi/3} 2a\cos\frac{\theta}{2}\, d\theta$$

$$\therefore \text{ arc } ACB = 2\int_{0} 2a\cos\frac{\theta}{2}\,d\theta$$

$$= 4a\left[2\sin\frac{\theta}{2}\right]_0^{\pi/3}$$

$$= 8a\left[\sin\frac{\pi}{6} - 0\right]$$

$$= 8a\left[\frac{1}{2}\right]$$

arc $ACB = 4a$.

Now the total perimeter of cardioide $= 8a$.

$\therefore$ The line $4r = 3a\sec\theta$ bisects the perimeter of the cardioide.

Find the total length of the curve

$$x = \frac{y^3}{3} + \frac{1}{4y} \quad \text{from } y = 1 \text{ to } y = 2$$

$$S = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$