RECTIFICATION - PARAMETRIC CURVES

Monday, April 26, 2021 11:30 AM

LENGTH OF THE ARC OF A CURVE GIVEN IN PARAMETRIC FORM

- Length of the arc of a curve given in parametric form as $x = f_1(t)$ and $y = f_2(t)$ is
- $S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

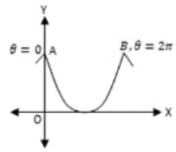
1) Find the length of one arc of the cycloid $x = a(\theta - sin\theta)$, $y = a(1 + cos\theta)$

$\theta = 0 \quad x = 0, \; y = 2a \qquad (0, 2a)$

$\theta = \pi \quad x = a\pi, \; y = 0 \qquad (a\pi, 0)$

$\theta = 2\pi \quad x = 2a\pi, \; y = 2a \qquad (2a\pi, 2a)$

$\theta = 0$ to $\theta = 2\pi$



The curve is shown here.

for A, $\theta = 0$ and for B, $\theta = 2\pi$

The required length

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = a(\theta - \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1 - \cos\theta)$

$y = a(1 + \cos\theta) \Rightarrow \frac{dy}{d\theta} = a(0 - \sin\theta)$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= a^2[2 - 2\cos\theta]$$

$$= 2a^2[1 - \cos\theta]$$

$$= 2a^2\left(2\sin^2\frac{\theta}{2}\right)$$

$$= 4a^2\sin^2\theta/2$$

$\therefore$ The required length $= \int_0^{2\pi} \sqrt{4a^2\sin^2\theta/2}\; d\theta$

$$= 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= 2a\left[\frac{-\cos\theta/2}{1/2}\right]_0^{2\pi}$$

$$= -4a[\cos\pi - \cos 0]$$

$$= -4a\left[\cos\pi - \cos 0\right]$$

$$= -4a\left[-1-1\right]$$

length of one arc of cycloid $= 8a$

Example 2) Prove that the length of the arc of the curve $x = a\sin 2\theta(1+\cos 2\theta), y = a\cos 2\theta(1-\cos 2\theta)$ measured from the origin to (x, y) is $\frac{4}{3}a\sin 3\theta$.

Soln:- $x = a\sin 2\theta(1+\cos 2\theta)$

$$\frac{dx}{d\theta} = 2a\cos 2\theta(1+\cos 2\theta) + a\sin 2\theta(0 - 2\sin 2\theta)$$

$$= 2a\cos 2\theta + 2a\cos^2 2\theta - 2a\sin^2 2\theta$$

$$= 2a\cos 2\theta + 2a(\cos^2 2\theta - \sin^2 2\theta)$$

$$\frac{dx}{d\theta} = 2a\cos 2\theta + 2a\cos 4\theta$$

$$y = a\cos 2\theta(1-\cos 2\theta)$$

$$\frac{dy}{d\theta} = -2a\sin 2\theta(1-\cos 2\theta) + a\cos 2\theta(0 + 2\sin 2\theta)$$

$$= -2a\sin 2\theta + 2a\sin 2\theta\cos 2\theta + 2a\sin 2\theta\cos 2\theta$$

$$= -2a\sin 2\theta + 4a\sin 2\theta\cos 2\theta$$

$$\frac{dy}{d\theta} = -2a\sin 2\theta + 2a\sin 4\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left[2a\cos 2\theta + 2a\cos 4\theta\right]^2 + \left[-2a\sin 2\theta + 2a\sin 4\theta\right]^2$$

$$= 4a^2\cos^2 2\theta + 4a^2\cos^2 4\theta + 8a^2\cos 2\theta\cos 4\theta + 4a^2\sin^2 2\theta + 4a^2\sin^2 4\theta - 8a^2\sin 2\theta\sin 4\theta$$

$$= 4a^2 + 4a^2 + 8a^2\left[\cos 2\theta\cos 4\theta - \sin 2\theta\sin 4\theta\right]$$

$$= 8a^2 + 8a^2\cos 6\theta = 8a^2(1+\cos 6\theta)$$

$$= 8a^2(2\cos^2 3\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 16a^2\cos^2 3\theta$$

$$\therefore \text{The required length} = \int^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$\therefore$ The required length $= \int_0^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{\theta} \sqrt{16a^2 \cos^2 3\theta}\, d\theta$$

$$= 4a \int_0^{\theta} \cos 3\theta \, d\theta$$

$$= 4a \left[\frac{\sin 3\theta}{3} \right]_0^{\theta}$$

$$= \frac{4a}{3} \left[\sin 3\theta - \sin 0 \right]$$

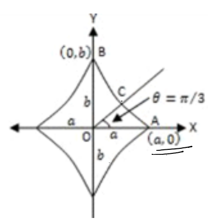
$$= \frac{4a}{3} \sin 3\theta$$

3) Find the total length of the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$. Hence, deduce the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. Also show that the line $\theta = \pi/3$ divides the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ in the first quadrant in the ratio 1: 3 .

Soln :- The curve is astroid as shown in the figure

1) To rectify the curve

$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$, Lets use the parametric equation of the curve.



The eqns are

$$x = a\cos^3\theta, \quad y = b\sin^3\theta$$

If s is the total length

$$s = 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = a\cos^3\theta \qquad \frac{dx}{d\theta} = -3a\cos^2\theta \sin\theta$

$y = b\sin^3\theta \qquad \frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9\left(a^2\cos^4\theta\sin^2\theta + b^2\sin^4\theta\cos^2\theta\right)$$

$$= 9\sin^2\theta\cos^2\theta\,(a^2\cos^2\theta + b^2\sin^2\theta)$$

$$\therefore S = 4\int_0^{\pi/2} \sqrt{9\sin^2\theta\cos^2\theta\,(a^2\cos^2\theta + b^2\sin^2\theta)}\,d\theta \quad \text{——} (1)$$

$$= 4\int_0^{\pi/2} 3\sin\theta\cos\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

put $a^2\cos^2\theta + b^2\sin^2\theta = t^2$

$$(-a^2\, 2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\,d\theta = 2t\,dt$$

$$2\sin\theta\cos\theta\,(b^2-a^2)\,d\theta = 2t\,dt$$

$$\sin\theta\cos\theta\,d\theta = \frac{t}{b^2-a^2}\,dt$$

when $\theta = 0 \Rightarrow t = a$

$\theta = \pi/2 \Rightarrow t = b$

$$\therefore S = 12\int_a^b \sqrt{t^2}\cdot\frac{t}{b^2-a^2}\,dt = \frac{12}{b^2-a^2}\int_a^b t^2\,dt$$

$$= \frac{12}{b^2-a^2}\left(\frac{t^3}{3}\right)_a^b = \frac{4}{b^2-a^2}(b^3-a^3)$$

$$S = \frac{4(a^2+ab+b^2)}{a+b}$$

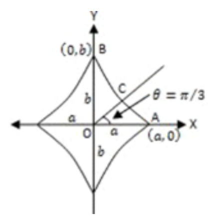
2) For second part, we just put $a=b$ to get length of $x^{2/3}+y^{2/3} = a^{2/3}$

$$S = \frac{4(a^2+a^2+a^2)}{a+a} = \frac{12a^2}{2a} = 6a.$$

(iii) For the third part,

the length of the astroid in the first quadrant

$$AB = \frac{S}{4} = \frac{6a}{4} = \frac{3}{2}a$$



Now the length cut off by $\theta = \pi/3$

$$\text{arc } AC = \int_0^{\pi/3} \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta \quad (\text{put } a=b \text{ in } (1))$$

$$= \int_0^{\pi/3} 3a \sin\theta \cos\theta \, d\theta$$

put $\sin\theta = t$
$\cos\theta \, d\theta = dt$

$$= \int_0^{\sqrt{3}/2} 3a \, t \, dt \quad = 3a \left(\frac{t^2}{2}\right)_0^{\sqrt{3}/2}$$

$$\text{arc } AC = \frac{3a}{2}\left(\frac{3}{4}\right) = \frac{9a}{8}$$

$$\therefore \text{arc } BC = \text{arc} AB - \text{arc} AC$$

$$= \frac{3a}{2} - \frac{9a}{8} = \frac{3a}{8}$$

$$\therefore \frac{\text{arc} BC}{\text{arc} AC} = \frac{3a/8}{9a/8} = \frac{1}{3}$$

$\therefore$ The line $\theta = \frac{\pi}{3}$ divides arc AB in the ratio 1:3

4) Show that the length of the tractrix , $x = a[\cos t + \log\tan(t/2)], y = a\sin t$ $from$ $t = \pi/2$ to any point t is $a log$ $(sin$ $t)$

Soln.: $x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \cdot \sec^2\left(\frac{t}{2}\right) \cdot \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$= a\left[\frac{1-\sin^2 t}{\sin t}\right] = \frac{a\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\therefore \frac{dy}{dt} = a\cos t$$

$$\text{Now } \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t$$

$$= a^2\cos^2 t\left[\frac{\cos^2 t}{\sin^2 t} + 1\right]$$

$$= a^2\cos^2 t\left(\cos^2 t + \sin^2 t\right)$$

$$\sin^2 t$$

$$= a^2 \cos^2 t \left[\frac{\cos^2 t + \sin^2 t}{\sin^2 t} \right]$$

$$= a^2 \frac{\cos^2 t}{\sin^2 t}$$

$\therefore$ The required length $= \displaystyle\int_{\pi/2}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_{\pi/2}^{t} a \frac{\cos t}{\sin t}\, dt$$

$$= a \left[\log \sin t \right]_{\pi/2}^{t}$$

$$= a \left[\log \sin t - \log \sin \pi/2 \right]$$

The required length $= a \log \sin t$