RECTIFICATION - CARTESIAN CURVES

Monday, April 19, 2021 11:30 AM

- Here, we shall be concerned with the determination of the length of arcs of plane curves whose equations are given in Cartesian, parametric or polar forms.
- The process is known as **rectification.**
- Hence **Rectification** means finding the length of the curve between two given points.

- **LENGTH OF THE ARC OF A CURVE GIVEN IN CARTESIAN FORM:**

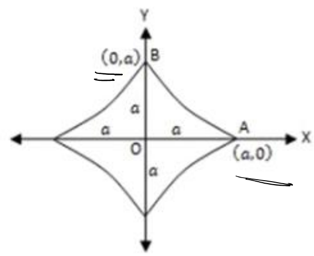
(i)Length of the arc of a curve given by $y = f(x)$ is $S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

(ii) Length of the arc of a curve given by $x = f(y)$ is $S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

Example 1) Find the total length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Sol:- The curve is called Four cusped hypocycloid or Astroid.

Its shape is shown in fig



$S = 4$ length from A to B

$$S = 4\int_a^0 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{— (1)}$$

Now, $x^{2/3} + y^{2/3} = a^{2/3}$

differentiating wrt x

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

Substituting in (1)

$$S = 4\int^0 \sqrt{1+\left(\frac{y^{2/3}}{\quad}\right)}\, dx$$

$$S = 4\int_a^0 \sqrt{1+\left(\frac{y^{2/3}}{x^{2/3}}\right)}\, dx$$

$$= 4\int_a^0 \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{2/3}}}\, dx$$

$$= 4\int_a^0 \sqrt{\frac{a^{2/3}}{x^{2/3}}}\, dx = 4\int_a^0 a^{1/3}\cdot x^{-1/3}\, dx$$

$$\therefore S = 4a^{1/3}\left[\frac{x^{2/3}}{2/3}\right]_a^0 = 4a^{1/3}\cdot\frac{3}{2}\left[0 - a^{2/3}\right]$$

$$S = -6a$$

$\therefore$ Total length of the curve $= 6a$.

Example 2) Find the length of the arc of $y = e^x$ from $(0,1) to\ (1,e)$

Soln:- The length of required arc $= \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$\because y = e^x \qquad \therefore \frac{dy}{dx} = e^x$

$$\therefore S = \int_0^1 \sqrt{1+e^{2x}}\, dx$$

put $e^x = t \qquad \therefore e^x dx = dt \Rightarrow dx = \frac{dt}{e^x}$

$dx = \frac{dt}{t}$

$x=0,\ t=1 \mid x=1,\ t=e$

$$S = \int_1^e \sqrt{1+t^2}\, dt$$

$$S = \int_1^e \sqrt{1+t^2}\ \frac{dt}{t}$$

$$= \int_1^e \frac{\sqrt{1+t^2}}{t}\, dt = \int_1^e \frac{1+t^2}{t\sqrt{1+t^2}}\, dt$$

$$= \int_1^e \frac{1}{t\sqrt{1+t^2}}\, dt + \int_1^e \frac{t}{\sqrt{1+t^2}}\, dt$$

In the first integral, put $\frac{1}{t} = u$ $\therefore -\frac{dt}{t^2} = du$

$$dt = -t^2 du = -\frac{1}{u^2} du$$

when $t = 1$, $u = 1$, when $t = e$, $u = \frac{1}{e}$

$$S = \int_1^{1/e} \frac{u}{\sqrt{1+\frac{1}{u^2}}} \cdot -\frac{1}{u^2} du + \int_1^e \frac{t}{\sqrt{1+t^2}}\, dt$$

$$= \int_1^{1/e} \frac{-du}{\sqrt{1+u^2}} + \int_1^e \frac{t}{\sqrt{1+t^2}}\, dt$$

$$= -\left[\log\left(u + \sqrt{u^2+1}\right)\right]_1^{1/e} + \left[\sqrt{1+t^2}\right]_1^e$$

$$= -\left[\left(\log\left(\frac{1}{e} + \sqrt{\frac{1}{e^2}+1}\right) - \log\left(1+\sqrt{2}\right)\right] + \left[\sqrt{1+e^2} - \sqrt{2}\right]$$

$$S = \sqrt{1+e^2} - \sqrt{2} - \log\frac{\left(1+\sqrt{1+e^2}\right)}{e} + \log\left(1+\sqrt{2}\right)$$

$$S = \sqrt{1+e^2} - \sqrt{2} - \log\left[\frac{1+\sqrt{1+e^2}}{e\left(1+\sqrt{2}\right)}\right]$$

Example 3) Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$

Soln :- The circle can be written as

$$x^2+y^2-6y+9 = 9$$

$$x^2+(y-3)^2 = 3^2$$

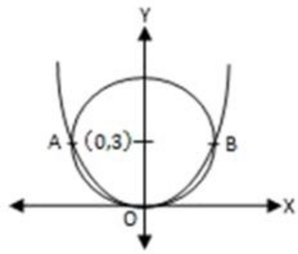
Hence, its centre is $(0,3)$ and radius is 3

The parabola $x^2=6y$ is symmetrical about the y-axis

The two curves

$$\underline{x^2+y^2=6y}, \quad \underline{x^2=4y}$$

will intersect where



$$4y+y^2=6y$$

$$y^2=2y$$

$$y(y-2)=0$$

$$y=0, \quad y=2$$

when $y=0$, $x=0$, when $y=2$, $x^2=4y=8$

$$x = \pm 2\sqrt{2}$$

$A(-2\sqrt{2}, 2)$, $B(2\sqrt{2}, 2)$

Required length $= AB = 2OB$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Now for parabola, $x^2=4y \Rightarrow y=\frac{x^2}{4}$

$$\therefore \frac{dy}{dx} = \frac{x}{2}$$

$$\therefore S = 2\int^{2\sqrt{2}} \sqrt{1+\frac{x^2}{}}\, dx.$$

$$\therefore \; S = 2\int_{0} \sqrt{1+\frac{x^2}{4}}\, dx$$

$$= \int_{0}^{2\sqrt{2}} \sqrt{x^2+4}\, dx$$

$$\left[\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)\right]$$

$$S = \left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log\left(x+\sqrt{x^2+4}\right)\right]_{0}^{2\sqrt{2}}$$

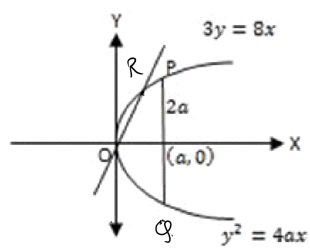
$$= \sqrt{2}\cdot\sqrt{12} + 2\log\left(2\sqrt{2}+\sqrt{12}\right) - 2\log 2$$

$$S = 2\left[\sqrt{6} + \log\left(\sqrt{2}+\sqrt{3}\right)\right]$$

Example 4) Show that the length of the parabola $y^2 = 4ax$ from the vertex to the end of the latus rectum is $a\left[\sqrt{2} + log\left(1+\sqrt{2}\right)\right]$. Find the length of arc cut off by the line $3y = 8x$

The latus rectum is line $x = a$

The end of latus rectum ie P is $(a, 2a)$



The required length = arc OP

$$= S = \int_{0}^{2a} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

we have $y^2 = 4ax \Rightarrow x = \frac{y^2}{4a} \Rightarrow \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

$$\therefore \quad s = \int_0^{2a} \sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{y^2+4a^2}\, dy$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$$

$$\therefore \quad s = \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a}{2}\sqrt{4a^2+4a^2} + \frac{4a^2}{2}\log\left(2a+\sqrt{4a^2+4a^2}\right) - \frac{4a^2}{2}\log\left(\sqrt{4a^2}\right)\right]$$

$$= \frac{1}{2}\sqrt{8a^2} + a\log\left(2a+\sqrt{8a^2}\right) - a\log(2a)]$$

$$= \sqrt{2}\,a + a\log\left(\frac{2a+2\sqrt{2}a}{2a}\right)$$

$$= a\left[\sqrt{2} + \log\left(1+\sqrt{2}\right)\right]$$

For the second part, Let the point of intersection for line $3y=8x$ and the parabola $y^2=4ax$ be R

$$\Rightarrow \quad x = \frac{3y}{8}$$

Sub in $\quad y^2 = 4ax = 4a\left(\frac{3y}{8}\right)$

$$y^2 = \frac{3a}{2}y$$

$$2y^2 - 3ay = 0 \Rightarrow y(2y-3a) = 0$$

$$\Rightarrow y = 0 \quad \text{or} \quad y = \frac{3a}{2}$$

$y = 0 \Rightarrow x = 0$

$y = \frac{3a}{2} \Rightarrow x = \frac{3y}{8} \Rightarrow \frac{9a}{16}. \qquad \therefore R \equiv \left(\frac{9a}{16}, \frac{3a}{2} \right)$

Length of the arc cut off by line $3y = 8x$

$$= \int_0^R \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^{3a/2} \sqrt{1 + \frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a} \left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2} \log\left(y + \sqrt{y^2+4a^2}\right) \right]_0^{3a/2}$$

(from first part)

$$= \frac{1}{2a} \left[\frac{3a}{4}\sqrt{\frac{9a^2}{4}+4a^2} + \frac{4a^2}{2}\log\left(\frac{3a}{2} + \sqrt{\frac{9a^2}{4}+4a^2}\right) - \frac{4a^2}{2}\log(2a) \right]$$

$$= \frac{1}{2a} \left[\frac{3a}{4}\left(\frac{5a}{2}\right) + \frac{4a^2}{2}\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - 2a^2\log(2a) \right]$$

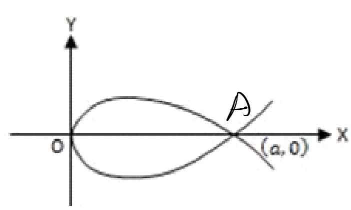
$$= \frac{1}{2a} \left[\frac{15a^2}{8} + 2a^2\log(4a) - 2a^2\log(2a) \right]$$

$$= \frac{15a}{16} + a\log\left(\frac{4a}{2a}\right) = \frac{15a}{16} + a\log 2.$$

Example 5) Find the length of the loop of the curve $3ay^2 = x(x-a)^2$

The curve cuts the x-axis at $O(0,0)$ and $A(a,0)$.

$\therefore$ The length of the loop is given by



$$S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

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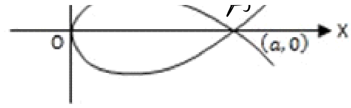


The given curve is $3ay^2 = x(x-a)^2$

differentiating wrt x

$$6ay\frac{dy}{dx} = (x-a)^2 + x\cdot 2(x-a)$$

$$= (x-a)\left(3x-a\right)$$

$$\frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

but $y^2 = \dfrac{x(x-a)^2}{3a}$ [from eqn of the curve]

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2(3x-a)^2}{36a^2} \times \frac{3a}{x(x-a)^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{12ax}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{(3x-a)^2}{12ax} = \frac{(3x+a)^2}{12ax}$$

$$\therefore S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^a \frac{3x+a}{2\sqrt{3}\,a\cdot\sqrt{x}}\, dx$$

$$= 2 \int^a$$

$$= \frac{2}{2\sqrt{3}a}\int_0^a \left(3\sqrt{x} + \frac{a}{\sqrt{x}}\right) dx$$

$$= \frac{2}{2\sqrt{3}a}\left[3 \cdot \frac{x^{3/2}}{3/2} + a \cdot \frac{x^{1/2}}{1/2}\right]_0^a$$

$$= \frac{1}{\sqrt{3}a}\left[2x^{3/2} + 2a\sqrt{x}\right]_0^a$$

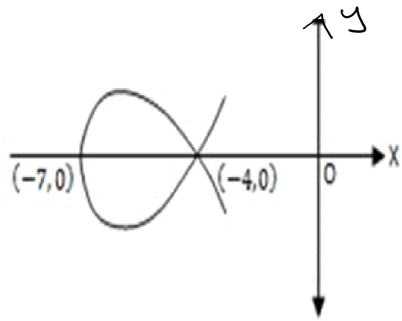
$$= \frac{2}{\sqrt{3}a}\left[a^{3/2} + a^{3/2} - 0\right] = \frac{4a}{\sqrt{3}}$$

Example 6) Find the total length of the loop of the curve $9y^2 = (x+7)(x+4)^2$

Soln:- If $y=0$, $x=-7$ or $x=-4$

the loop intersects the x-axis at

$x=-7$ and $x=-4$



If s is the total length of the loop

$$s = 2\int_{-7}^{-4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$9y^2 = (x+7)(x+4)^2$$

differentiating wrt x

$$18y\frac{dy}{dx} = (x+4)^2 + (x+7)\cdot 2(x+4)$$

$$= (x+4)(2x+14+x+4)$$

$$\frac{dy}{dx} = \frac{3(x+4)(x+6)}{18y} = \frac{(x+4)(x+6)}{6y}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+4)^2(x+6)^2}{36y^2}$$

Now $9y^2 = (x+7)(x+4)^2$

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x+4)^2\,(x+6)^2}{4\,(x+7)(x+4)^2} = \frac{(x+6)^2}{4(x+7)}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{(x+6)^2}{4(x+7)} = \frac{(x+8)^2}{4(x+7)}$$

$$\therefore\ S = 2\int_{-7}^{-4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_{-7}^{-4} \frac{x+8}{2\sqrt{x+7}}\, dx$$

put $x+7 = t^2$ $\qquad dx = 2t\,dt$

$x = -7,\ t = 0,\quad$ when $x = -4,\ t = \sqrt{3}$

$$S = 2\int_{0}^{\sqrt{3}} \frac{t^2+1}{2t}\cdot 2t\,dt$$

$$= 2\int_{0}^{\sqrt{3}} (t^2+1)\,dt = 2\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}}$$

$$= 2\left[\sqrt{3}+\sqrt{3}\right]$$

$$\boxed{S = 4\sqrt{3}}$$