GAMMA FUNCTION

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DEFINITION

The function of n $(n>0)$ defined by the integral

$\int_0^\infty e^{-x} x^{n-1} dx$ is called Gamma Function and is denoted by Γn.

Thus $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Sometimes we may write it as $\int_0^\infty e^{-x} x^n dx = \Gamma n+1$

PROPERTIES OF GAMMA FUNCTION:

$\Gamma n+1 = n\Gamma n$ or $\Gamma n = (n-1)\Gamma n-1$

Proof :- $\Gamma n+1 = \int_0^\infty e^{-x} x^n dx$

Integrate by parts

$$\Gamma n+1 = \left[x^n\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty n x^{n-1}\left(-e^{-x}\right) dx$$

but $\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0$

(By L'Hospital's Rule)

$$\therefore \Gamma n+1 = (0-0) + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$\boxed{\Gamma n+1 = n\Gamma n}$$

Values of Γn :

1) If n is a positive integer, by successive application of above property

$$\Gamma(n+1) = n\Gamma n = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2) \cdots\cdots 1\,\Gamma 1$$

Now $\Gamma 1 = \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx = \left(-e^{-x}\right)_0^\infty$

$$= -\left(e^{-\infty} - e^0\right) = 1$$

$$\Gamma(n+1) = n(n-1)(n-2) \cdots\cdots 1 = n!$$

$\Gamma(n+1) = n!$ if n is a positive integer

For this reason, Gamma Function is often referred to as generalised factorial.

For eg. $\Gamma 6 = 5! = 120$

$\Gamma 9 = 8! =$

$\Gamma 2 = 1! = 1$

(ii) $\Gamma 0 = \dfrac{\Gamma 1}{0} = \dfrac{1}{0} = \infty$

$\Gamma(n+1) = n\Gamma n$

$\Rightarrow \Gamma n = \dfrac{\Gamma(n+1)}{n}$

(iii) If n is a negative integer.

By making use of $\Gamma n = \frac{\Gamma n+1}{n}$ repeatedely, we will get

$$\Gamma -5 = \frac{\Gamma -4}{(-5)} = \frac{\Gamma -3}{(-5)(-4)} = \frac{\Gamma -2}{(-5)(-4)(-3)} = \frac{\Gamma -1}{(-5)(-4)(-3)(-2)}$$

$$= \frac{\Gamma 0}{(-5)(-4)(-3)(-2)(-1)} = \infty$$

(iv) If n is a positive fraction, $(n>1)$

we can use $\Gamma n+1 = n\Gamma n$ repeatebly and find Γn in terms of Γx where $0<x<1$

For ex: $\Gamma \frac{5}{2} = \frac{3}{2}\Gamma \frac{3}{2} = \frac{3}{2}\left(\frac{1}{2}\right)\Gamma \frac{1}{2}$ $\Gamma n+1 = n\Gamma n$

(v) If n is negative fraction,

use $\Gamma n = \frac{\Gamma n+1}{n}$ repeatebly and find Γn in terms of Γx where $0<x<1$

$$\Gamma \frac{-5}{3} = \frac{\Gamma \frac{-5}{3}+1}{\left(\frac{-5}{3}\right)} = \frac{\Gamma \frac{-2}{3}}{\left(\frac{-5}{3}\right)} = \frac{\Gamma \frac{-2}{3}+1}{\left(\frac{-5}{3}\right)\left(\frac{-2}{3}\right)} = \frac{\Gamma \frac{1}{3}}{\left(\frac{-5}{3}\right)\left(\frac{-2}{3}\right)}$$

(vi) If $0<n<1$, we find Γn by numerical Integration

Thus Γn is defined for all n, except $n = 0, -1, -2, -3, \ldots$

Second Form of Gamma Function :-

$$\overline{|n} = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx$$

Proof :- $\overline{|n} = \int_0^\infty e^{-x} x^{n-1} dx$

put $x = t^2$ $\quad dx = 2t\, dt$

$$\overline{|n} = \int_0^\infty e^{-t^2} (t^2)^{n-1} \cdot 2t\, dt$$

$$= 2\int_0^\infty e^{-t^2}\, t^{2n-2} \cdot t\, dt$$

$$\overline{|n} = 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\, dt$$

$$\overline{|n} = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1}\, dx$$ using dummy variable property

Types of examples on Gamma Functions :-

Type-I :- Evaluate $\int_0^\infty e^{-ax^n} dx$

Method :- $ax^n = t$

Type-II :- Evaluate $\int^\infty x^m \cdot e^{-ax^n} dx$

Type – II :- Evaluate $\int_0^\infty x^m \cdot e^{-ax^n}\, dx$

Method :- $ax^n = t$

Type – III :- Evaluate $\int_0^1 x^m (\log x)^n\, dx$

Method :- put $\log x = -t$

Type – IV :- Evaluate $\int_0^\infty \frac{x^a}{a^x}\, dx$

Method :- put $a^x = e^t$

Ex-1 :- Given $\Gamma(1\cdot8) = 0\cdot9314$, find the value of $\Gamma(-2\cdot2)$

Soln :- we have $\Gamma(n+1) = n\,\Gamma(n)$

$$\therefore \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\therefore \Gamma(-2\cdot2) = \frac{\Gamma(-2\cdot2+1)}{(-2\cdot2)} = \frac{\Gamma(-1\cdot2)}{(-2\cdot2)} = \frac{\Gamma(-1\cdot2+1)}{(-2\cdot2)(-1\cdot2)} = \frac{\Gamma(-0\cdot2)}{(-2\cdot2)(1\cdot2)}$$

$$= \frac{\Gamma(-0\cdot2+1)}{(-2\cdot2)(-1\cdot2)(-0\cdot2)} = \frac{\Gamma(0\cdot8)}{(-2\cdot2)(-1\cdot2)(-0\cdot2)} = \frac{\Gamma(1\cdot8)}{(-2\cdot2)(-1\cdot2)(-0\cdot2)(0\cdot8)}$$

$$\Gamma(-2\cdot2) = \frac{0\cdot9314}{} \qquad - \quad -2\cdot21$$

$$\Gamma(-2.2) = \frac{0.9314}{(-2.2)(-1.2)(-0.2)(0.8)} = -2.21$$

Ex 2:- Prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2^n}\sqrt{\pi}$.

Hence or otherwise prove that $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!\,4^n}$

Soln:- clearly n must be a positive integer

$$\therefore \Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\Gamma\left(n-\tfrac{1}{2}\right) \qquad \left(\Gamma n = (n-1)\Gamma(n-1)\right)$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\Gamma\left(n-\tfrac{3}{2}\right)$$

$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\Gamma\left(n-\tfrac{5}{2}\right) \quad \text{and so on}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\cdots\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\cdots\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{1\cdot3\cdot5\cdots\cdots(2n-1)\sqrt{\pi}}{2^n}$$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\cdots\cdots(2n-2)(2n-1)(2n)\sqrt{\pi}}{\quad}$$

$$2\cdot 4\cdot 6 \cdots\cdots (2n-2)(2n)\ 2^n$$

$$= \frac{(2n)!\sqrt{\pi}}{2^n\left(1\cdot 2\cdot 3\cdots\cdots(n-1)\cdot n\right)2^n}$$

$$\Gamma(n+1) = \frac{(2n)!\sqrt{\pi}}{n!\,4^n}$$

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Revision :- $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\,dx$

also $\Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\,dx$

Property :- $\Gamma(n+1) = n\Gamma n$

n is +ve integer $\rightarrow$ $\Gamma n = (n-1)!$

n is zero $\Gamma 0 = \infty$

n is -ve integer $\Gamma n = \infty$

n is fraction $\rightarrow$ Γn can be evaluated.

$0 < n < 1$ $\rightarrow$ Γn can be evaluated using numerical integration

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Ex-3 If $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\,\Gamma\frac{n+1}{2}}{\Gamma n}$, Show that $I_{n+2} = \dfrac{n+1}{n+2} I_n$

$$\frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, \quad \text{Show that } I_{n+2} = \frac{n+1}{n+2} I_n$$

and hence find I_5

Sol^n:-
$$I_n = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\Rightarrow \quad I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+2+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)}$$

$\Gamma(n+1) = n\,\Gamma(n)$ using this $\Gamma\left(\frac{n+3}{2}\right) = \Gamma\left(\frac{n+1}{2}+1\right) = \left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)$

$$\Gamma\left(\frac{n+2}{2}+1\right) = \left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right)$$

$$\therefore \quad I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\cdot\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right)} = \frac{(n+1)}{n+2}\cdot\frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$$\therefore \quad I_{n+2} = \frac{n+1}{n+2} I_n$$

To find I_5, put $n=3$

$$I_5 = \frac{3+1}{3+2} I_3 = \frac{4}{5} I_3 \quad \text{——— (1)}$$

using the relation again for $n=1$

$$I_3 = \frac{1+1}{1+2} I_1 = \frac{2}{3} I_1$$

$$\therefore I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \qquad (\text{using } (1))$$

$$\text{Now, } I_1 = \frac{\frac{\sqrt{\pi}}{2} \cdot \Gamma\left(\frac{1+1}{2}\right)}{\Gamma\left(\frac{1}{2}+1\right)} = \frac{\frac{\sqrt{\pi}}{2} \Gamma(1)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2} \cdot \sqrt{\pi}} = 1$$

$$\therefore I_5 = \frac{8}{15} \cdot (1) = \frac{8}{15}$$

Ex4 :- $\displaystyle\int_0^\infty e^{-h^2 x^2} dx$

Soln :- Put $h^2 x^2 = t$ $\quad \therefore x = \dfrac{\sqrt{t}}{h}$ $\quad \therefore dx = \dfrac{1}{h} \cdot \dfrac{1}{2\sqrt{t}} \cdot dt$

$$\therefore I = \int_0^\infty e^{-h^2 x^2} dx = \int_0^\infty e^{-t} \cdot \frac{1}{h} \cdot \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2h} \int_0^\infty e^{-t}\, t^{-1/2} dt$$

$$= \frac{1}{2h} \int_0^\infty e^{-t}\, t^{(1/2 - 1)} dt \qquad \left[\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \right]$$

$$= \frac{1}{2h} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2h}$$

Ex-5 $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$

Soln :- Let $I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx$

put $\sqrt{x} = t \rightarrow x = t^2 \longrightarrow dx = 2t\,dt$

$$\therefore I = \int_0^\infty (t^2)^{\frac{1}{4}} e^{-t}\, 2t\, dt = 2\int_0^\infty t^{1/2} e^{-t} . t\, dt$$

$$= 2\int_0^\infty e^{-t} t^{3/2}\, dt$$

$$= 2\,\Gamma\tfrac{5}{2}$$

$\Gamma{n+1} = n\Gamma{n}$

$$= 2\left(\tfrac{3}{2}\right)\Gamma\tfrac{3}{2}$$

$$= 2\left(\tfrac{3}{2}\right)\left(\tfrac{1}{2}\right)\Gamma\tfrac{1}{2}$$

$$I = \frac{3\sqrt{\pi}}{2}$$

Ex-6 :- Prove that $\int_0^\infty x e^{-x^8}\, dx \cdot \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$

Soln :- Let $I_1 = \int_0^\infty x e^{-x^8}\, dx$ and $I_2 = \int_0^\infty x^2 e^{-x^4}\, dx$

put $x^8 = t$ in I_1

$$x = t^{1/8} \longrightarrow dx = \frac{1}{8} t^{-7/8} dt$$

$$I_1 = \int_0^\infty t^{1/8} e^{-t} \cdot \frac{1}{8} t^{-7/8} dt = \frac{1}{8}\int_0^\infty e^{-t} t^{-3/4} dt$$

$$\therefore I_1 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)$$

In I_2, put $x^4 = t$

$$x = t^{1/4} \longrightarrow dx = \frac{1}{4} t^{-3/4} dt$$

$$\therefore I_2 = \int_0^\infty t^{1/2} e^{-t} \frac{1}{4} t^{-3/4} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{-1/4} dt$$

$$I_2 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore LHS = I_1 \cdot I_2 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4}\Gamma\left(\frac{3}{4}\right) = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\,\pi \qquad \left(\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}\right)$$

$$LHS = \frac{1}{32}\cdot\sqrt{2}\,\pi = \frac{\pi}{16\sqrt{2}} = RHS.$$

Ex-7 :- $\int_0^1 x^m \cdot \left(\log \frac{1}{x}\right)^n dx$

Soln :- $I = \int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \int_0^1 x^m (-\log x)^n \, dx$

$$= (-1)^n \int_0^1 x^m (\log x)^n \, dx$$

put $\log x = -t \rightarrow x = e^{-t}$

$dx = -e^{-t} dt$

$x = 0, \quad t = \infty$

$x = 1 \quad t = 0$

$$I = (-1)^n \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}) \, dt$$

$$I = (-1)^{2n} \int_\infty^0 (-1)(e^{-t})^{m+1} t^n \, dt$$

$$= \int_0^\infty (e^{-t})^{m+1} t^n \, dt$$

put $(m+1)t = u \rightarrow t = \frac{u}{m+1} \rightarrow dt = \frac{du}{m+1}$

$$I = \int^\infty e^{-u} \left(\frac{u}{}\right)^n \frac{du}{}$$

$$I = \int_0^\infty e^{-u}\left(\frac{u}{m+1}\right)^n \frac{du}{m+1}$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^n \, du$$

$$I = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

Ex 8 :- $\int_0^1 (x \log x)^3 \, dx$

Soln:- $I = \int_0^1 x^3 (\log x)^3 \, dx$

put $\log x = -t \rightarrow x = e^{-t} \rightarrow dx = -e^{-t} dt$

when $x = 0$, $t = \infty$, $x = 1$, $t = 0$

$$I = \int_\infty^0 (e^{-t})^3 (-t)^3 (-e^{-t}) \, dt$$

$$= (-1)^3 \int_0^\infty e^{-4t} t^3 \, dt$$

put $4t = u \rightarrow t = \frac{u}{4}$ $\quad dt = \frac{du}{4}$

$$I = -\int_0^\infty e^{-u} \left(\frac{u}{4}\right)^3 \frac{du}{4}$$

$$= \frac{-1}{4^4} \int_0^\infty e^{-u} u^3 \, du$$

$$= \frac{-1}{4^4} \Gamma 4 = \frac{-3!}{256} = \frac{-3}{128}$$

Ex-9 $\int_0^\infty \frac{x^7}{7^x} \, dx$

Soln:- $I = \int_0^\infty \frac{x^7}{7^x} \, dx$

put $7^x = e^t$

$\Rightarrow t = x \log 7 \quad \Rightarrow dt = (\log 7) dx \Rightarrow dx = \frac{dt}{\log 7}$

when $x = 0$, $t = 0 \log 7 = 0$

when $x = \infty$ $t = (\infty) \log 7 = \infty$

$$\therefore I = \int_0^\infty \left(\frac{t}{\log 7}\right)^7 \cdot e^{-t} \cdot \frac{dt}{\log 7}$$

$$\therefore I = \frac{1}{(\log 7)^8} \int_0^\infty e^{-t} t^7 \, dt = \frac{\Gamma 8}{(\log 7)^8} = \frac{7!}{(\log 7)^8}$$

Ex-10 : $\int_0^\infty 7^{-4x^2} \, dx$

Ex-10 :- $\int_0 \quad + \quad dx$

Sol^n :- $I = \int_0^\infty 7^{-4x^2}\, dx$

put $7^{-4x^2} = e^{-t}$

$$-4x^2 (\log 7) = -t \Rightarrow 4x^2 \log 7 = t$$

$$x^2 = \frac{t}{4\log 7} \Rightarrow x = \sqrt{\frac{t}{4\log 7}}$$

$$dx = \frac{1}{2\sqrt{\log 7}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

when $x=0,\ t=0$ and when $x=\infty,\ t=\infty$

$$\therefore I = \int_0^\infty e^{-t} \frac{1}{2\sqrt{\log 7}} \cdot \frac{1}{2\sqrt{t}}\, dt$$

$$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{4\sqrt{\log 7}} \cdot \sqrt{\tfrac{1}{2}} \quad =$$

$$I = \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$$

Ex-11 :- Show that

(i) $\displaystyle\int_0^\infty x^{m-1} \cos ax\, dx = \frac{\sqrt{m}}{a^m} \cos\left(\frac{m\pi}{2}\right)$

(i) $\int_0^\infty x^{m-1} \cos ax \, dx = \frac{\Gamma m}{a^m} \cos\left(\frac{m\pi}{2}\right)$

(ii) $\int_0^\infty x^{m-1} \sin ax \, dx = \frac{\Gamma m}{a^m} \sin\left(\frac{m\pi}{2}\right)$

$\left[e^{-iax} = \cos ax - i \sin ax\right]$

Sol^n :- $I = \int_0^\infty x^{m-1} e^{-iax} \, dx$

put $iax = t \rightarrow x = \frac{t}{ia} \rightarrow dx = \frac{dt}{ia}$

$$I = \int_0^\infty \left(\frac{t}{ia}\right)^{m-1} e^{-t} \frac{dt}{ia}$$

$$= \frac{1}{(ia)^m} \int_0^\infty e^{-t} t^{m-1} \, dt$$

$$I = \frac{\Gamma m}{a^m} \left(\frac{1}{i^m}\right)$$

Now $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

$$i^m = \cos \frac{m\pi}{2} + i \sin \frac{m\pi}{2}$$

$$\frac{1}{i^m} = \cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2}$$

$$I = \frac{\Gamma m}{a^m} \left[\cos\left(\frac{m\pi}{2}\right) - i \sin\left(\frac{m\pi}{2}\right)\right]$$

$$\int_0^\infty e^{-iax} x^{m-1} \, dx = \frac{\Gamma m}{a^m} \left[\cos\left(\frac{m\pi}{2}\right) - i \sin\left(\frac{m\pi}{2}\right)\right]$$

$$\int_0^\infty e^{-iax} x^{m-1}\, dx = \frac{\Gamma m}{a^m}\left[\cos\left(\frac{m\pi}{2}\right) - i\sin\left(\frac{m\pi}{2}\right)\right]$$

$$\int_0^\infty x^{m-1}\cos ax\, dx - i\int_0^\infty x^{m-1}\sin ax\, dx = \frac{\Gamma m}{a^m}\cos\left(\frac{m\pi}{2}\right) - i\,\frac{\Gamma m}{a^m}\sin\left(\frac{m\pi}{2}\right)$$

comparing real and imaginary parts we get the required answers.

Ex-12:- prove that $\displaystyle\int_0^\infty \cos(ax^{1/n})\, dx = \frac{\Gamma n+1}{a^n}\cos\left(\frac{n\pi}{2}\right)$

Soln:- $\displaystyle I = \int_0^\infty \cos(ax^{1/n})\, dx$

put $ax^{1/n} = t$

$$x^{1/n} = \frac{t}{a} \rightarrow x = \frac{t^n}{a^n} \rightarrow dx = \frac{n t^{n-1}}{a^n}\, dt$$

$$I = \int_0^\infty \cos t\, \frac{n t^{n-1}}{a^n}\, dt$$

$$= \frac{n}{a^n}\int_0^\infty t^{n-1}\cos t\, dt$$

$$= \text{R.P of } \frac{n}{} \int^\infty t^{n-1} e^{-it}\, dt$$

$$= R.P \text{ of } \frac{n}{a^n} \int_0 t^{n-1} e^{-it}\, dt$$

complete as the previous sum.

Ex-13 Prove that

(i) $\displaystyle\int_0^\infty x e^{-ax} \cos bx \, dx = \frac{a^2-b^2}{(a^2+b^2)^2}$

(ii) $\displaystyle\int_0^\infty x e^{-ax} \sin bx \, dx = \frac{2ab}{(a^2+b^2)^2}$

Soln:-

$$I = \int_0^\infty x e^{-ax} \cdot e^{ibx} \, dx$$

$$= \int_0^\infty x e^{-(a-ib)x} \, dx$$

put $(a-ib)x = t$ $\qquad x = \dfrac{t}{a-ib} \rightarrow dx = \dfrac{dt}{a-ib}$

$$I = \int_0^\infty e^{-t} \cdot \frac{t}{a-ib} \cdot \frac{dt}{a-ib}$$

$$= \frac{1}{(a-ib)^2} \int_0^\infty e^{-t}\, t \, dt$$

$$I = \frac{1}{(a-ib)^2} \Gamma 2 = \frac{1!}{(a-ib)^2} = \frac{1}{(a-ib)^2}$$

$$= \frac{1}{(a^2-b^2)-2iab} \cdot \frac{(a^2-b^2)+2iab}{(a^2-b^2)+2iab}$$

$$= \frac{(a^2-b^2)+i2ab}{\quad} = \frac{(a^2-b^2)+i2ab}{\quad 2}$$

$$= \frac{(a^2-b^2)+i2ab}{(a^2-b^2)^2+4a^2b^2} = \frac{(a^2-b^2)+i2ab}{(a^2+b^2)^2}$$

$$\int_0^\infty x e^{-(a-ib)x}\,dx = \frac{(a^2-b^2)}{(a^2+b^2)^2} + i\,\frac{2ab}{(a^2+b^2)^2}$$

$$\int_0^\infty x e^{-ax}\cos bx\,dx + i\int_0^\infty x e^{-ax}\sin bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2} + i\,\frac{2ab}{(a^2+b^2)^2}$$

comparing real and imaginary parts,
we get the answer.