DIFFERENTIATION UNDER INTEGRAL SIGN

- Many a times we use the standard rules of integral calculus for evaluating some of the definite integrals.
- However, in certain cases where the standard rules do not work, the concept of differentiation under integral sign is used for evaluation of some of the definite integrals.
- If the function under integral sign satisfies certain conditions, then we can differentiate the given function under the integral sign and from the resulting function we can obtain the required integral.
- This is known as **differentiation under integral sign** abbreviated as **D.U.I.S.**

RULE

- If $f(x, \alpha)$ is a continuous function of x, and α is a parameter and
- if $\partial f / \partial \alpha$ is a continuous function of x and α together throughout the interval $[a, b]$ where a, b are constant and independent of α and
- if $I(\alpha) = \int_a^b f(x, \alpha) dx$ then $\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha} dx$

1) P.T $\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1 + \alpha), \alpha \geq 0$ Hence, evaluate $\int_0^1 \frac{x^7 - 1}{\log x} dx$

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx \qquad \text{———} \quad (1)$$

By the rule of DUIS, differentiating wrt α

$$\frac{dI}{d\alpha} = \int_0^1 \frac{1}{\log x} \left[x^\alpha \log x \right] dx = \int_0^1 x^\alpha dx$$

$$\frac{dI}{d\alpha} = \int_0^1 \frac{1}{\log x}\left[x^{\alpha}\log x\right] dx = \int_0^1 x^{\alpha}\, dx$$

$$\therefore \frac{dI}{d\alpha} = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1} - 0 = \frac{1}{\alpha+1}$$

$$\therefore \quad dI = \left(\frac{1}{\alpha+1}\right) d\alpha$$

Integrating both sides

$$I = \int \frac{1}{\alpha+1}\, d\alpha$$

$$\boxed{I(\alpha) = \log(\alpha+1) + C}$$

put $\alpha = 0$

$$I(0) = \log(1) + C$$

$$\therefore C = I(0)$$

To get value of $I(0)$, put $\alpha = 0$ in (1)

$$I(0) = \int_0^1 \frac{x^0 - 1}{\log x}\, dx = 0$$

$$\therefore C = 0$$

$$\therefore I(\alpha) = \log(\alpha+1)$$

ie $$\boxed{\int_0^1 \frac{x^{\alpha}-1}{\log x}\, dx = \log(\alpha+1)}$$

put $\alpha = 7$

$$\therefore \int_0^1 \frac{x^7-1}{\log x}\,dx = \log(7+1) = \boxed{\log 8}$$

2) PROVE THAT $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx = \log(1+a)$, where $a > -1$

Sol^n :- let $I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(1-e^{-ax}\right)dx$ ——— (1)

By DUIS, differentiating wrt a

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}\left[0 - e^{-ax}(-x)\right]dx$$

$$= \int_0^\infty \frac{e^{-x}}{x}\left(x e^{-ax}\right)dx = \int_0^\infty e^{-(1+a)x}\,dx$$

$$= \left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty = \left[0 - \frac{1}{-(1+a)}\right]$$

$$\therefore \frac{dI}{da} = \frac{1}{1+a}$$

$$dI = \frac{1}{1+a}\,da$$

Integrating both sides

$$I = \int \frac{1}{1+a}\,da = \log(1+a) + C$$

$$I = \int \frac{1}{1+a}\,da = \log(1+a) + c$$

$$\boxed{I(a) = \log(1+a) + c}$$

put $a = 0$, $\quad I(0) = \log(1) + c$

$$\therefore c = I(0)$$

but from (1), $\quad I(0) = \int_0^\infty \frac{e^{-x}}{x}(1-1)\,dx = 0$

$$\therefore \quad c = 0$$

$$\therefore I(a) = \log(1+a)$$

$$\int_0^\infty \frac{e^{-x}}{x}\left(1 - e^{-ax}\right) dx = \log(1+a)$$

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3) Prove that $\int_0^\infty e^{-ax} \cdot \frac{\sin mx}{x}\,dx = tan^{-1}\left(\frac{m}{a}\right)$ (a is a parameter) Given: $\int_0^\infty \frac{\sin mx}{x}\,dx = \frac{\pi}{2}$

Soln: Let $I(a) = \int_0^\infty e^{-ax}\,\frac{\sin mx}{x}\,dx$ —— (1)

Using DUIS, differentiating wrt a

$$\frac{dI}{da} = \int_0^\infty \frac{\sin mx}{x}\left[e^{-ax}(-x)\right] dx$$

$$= -\int_0^\infty \sin mx \, e^{-ax} \, dx$$

$$\therefore \frac{dI}{da} = -\left[\frac{e^{-ax}}{a^2+m^2}\left(-a\sin mx - m\cos mx\right)\right]_0^\infty$$

$$\boxed{\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$$

$$\frac{dI}{da} = \frac{-1}{a^2+m^2}\left[0-\left(-a(0)-m(1)\right)\right]$$

$$\frac{dI}{da} = \frac{-m}{a^2+m^2}$$

$$dI = \frac{-m}{a^2+m^2}\,da$$

Integrating both sides wrt a

$$I(a) = \int \frac{-m}{a^2+m^2}\,da = -m \cdot \frac{1}{m}\tan^{-1}\left(\frac{a}{m}\right)+c$$

$$\boxed{I(a) = -\tan^{-1}\left(\frac{a}{m}\right)+c}$$

put $a = 0$

$$I(0) = -\tan^{-1}\left(\frac{0}{m}\right) + c$$

$$\therefore \quad c = I(0)$$

$$\text{but } I(0) = \int_0^\infty \frac{\sin mx}{x}\,dx \qquad (\text{from } ③)$$

$$= \frac{\pi}{2} \qquad (\text{from given data})$$

$$\therefore \quad c = \frac{\pi}{2}$$

$$\therefore \ I(a) = -\tan^{-1}\left(\frac{a}{m}\right) + \frac{\pi}{2}$$

$$= \cot^{-1}\left(\frac{a}{m}\right)$$

$$\boxed{I(a) = \tan^{-1}\left(\frac{m}{a}\right)}$$

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4) Prove that $\int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right) dx = (1+a)\log(1+a) - a$

Sol^n

$$\text{let } I(a) = \int_0^\infty \frac{e^{-x}}{x}\left(a - \frac{1}{x} + \frac{1}{x}e^{-ax}\right) dx \quad \text{——} ①$$

By DUIS, Differentiating wrt a

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}\left(1 - 0 + \frac{1}{x}\left(e^{-ax}\cdot(-x)\right)\right) dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{e^{-x}}{x}\left(1 - e^{-ax}\right) dx \quad \text{—— ②}$$

Apply DUIS again, differentiating wrt a

$$\frac{d^2I}{da^2} = \int_0^\infty \frac{e^{-x}}{x}\left(0 - e^{-ax}(-x)\right) dx$$

$$= \int_0^\infty e^{-(1+a)x}\, dx = \left[\frac{e^{-(1+a)x}}{-(1+a)}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-(1+a)}\right] = \frac{1}{1+a}$$

$$\boxed{\frac{d^2I}{da^2} = \frac{1}{1+a}}$$

Integrate wrt a

$$\frac{dI}{da} = \int \frac{1}{1+a}\, da = \log(1+a) + c$$

put $a = 0$

$$\frac{dI}{da}(0) = \log(1) + c \Rightarrow c = \frac{dI}{da}(0)$$

da

Using (2) $\frac{dI}{da}(0) = \int_0^\infty \frac{e^{-x}}{x}(1-1)\,dx = 0$

$\therefore\ C = 0$

$$\therefore \boxed{\frac{dI}{da} = \log(1+a)}$$

Integrating again wrt a

$$I(a) = \int \log(1+a)\,da$$

Integrating by parts

$$= \log(1+a)\int 1\,da - \int \frac{1}{1+a}\cdot(a)\,da$$

$$= a\log(1+a) - \int \frac{a}{1+a}\,da$$

$$= a\log(1+a) - \left[\int 1\,da - \int \frac{1}{1+a}\,da\right]$$

$$= a\log(1+a) - \left[a - \log(1+a)\right] + C_1$$

$$I(a) = a\log(1+a) - a + \log(1+a) + C_1$$

To find C1, put $a=0$

$$I(0) = \log(1) + C_1 \Rightarrow C_1 = I(0)$$

put $a = 0$ in ① we get $I(0) = 0$

$$\therefore C_1 = 0$$

$$\therefore I(a) = a\log(1+a) - a + \log(1+a)$$

$$I(a) = (1+a)\log(1+a) - a$$

5) Prove that $\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx = \pi \sin^{-1} a, \quad 0 \le a < 1$

Soln :- $$I(a) = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx \quad \text{——— ①}$$

By DUIS, Differentiating wrt a

$$\frac{dI}{da} = \int_0^{\pi} \frac{1}{\cos x} \cdot \frac{1}{1+a\cos x} \cdot \cos x \, dx$$

$$\frac{dI}{da} = \int_0^{\pi} \frac{1}{1+a\cos x}\,dx$$

put $\tan\frac{x}{2} = t$, $dx = \frac{2\,dt}{1+t^2}$

$$\cos x = \frac{1-t^2}{1+t^2}$$

when $x = 0$, $t = 0$, when $x = \pi$, $t = \infty$

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{(\ldots)} \cdot \frac{2\,dt}{\ldots}$$

$$\frac{dI}{da} = \int_0^\infty \frac{1}{1+a\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int_0^\infty \frac{1+t^2}{(1+t^2)+a(1-t^2)} \cdot \frac{2\,dt}{1+t^2}$$

$$= \int_0^\infty \frac{2\,dt}{(1+a)+(1-a)t^2} = \frac{1}{1-a}\int_0^\infty \frac{2\,dt}{\left(\frac{1+a}{1-a}\right)+t^2}$$

$$\left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right.$$

$$= \frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}\left[\tan^{-1}\sqrt{\frac{1-a}{1+a}}\,t\right]_0^\infty$$

$$= \frac{2}{\sqrt{1-a^2}}\left[\frac{\pi}{2}\right] = \frac{\pi}{\sqrt{1-a^2}}$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}$$

Integrating wrt a

$$I(a) = \int \frac{\pi}{\sqrt{1-a^2}}\,da = \pi\sin^{-1}a + c$$

To find c, put $a=0$

$$I(0) = \pi\sin^{-1}(0) + c$$

$\Rightarrow c = I(0)$

from ①, $I(0) = 0$

$\therefore c = 0$

$\therefore \boxed{I(a) = \pi \sin^{-1}(a)}$

6) Prove that $\int_0^{\pi/2} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx = \pi\left[\sqrt{a+1} - 1\right], a > -1.$

- Let $I(a) = \int_0^{\pi/2} \frac{\log(1+a\sin^2 x)}{\sin^2 x} dx$
- By the rule of differentiation under the integral sign
- $\therefore \frac{dI}{da} = \int_0^{\pi} \frac{\partial f}{\partial a} dx = \int_0^{\pi/2} \frac{1}{1+a\sin^2 x} \cdot \frac{\sin^2 x}{\sin^2 x} dx$
- $= \int_0^{\pi/2} \frac{1}{1+a\sin^2 x} dx$ [Dividing by $\cos^2 x$]
- $= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + a\tan^2 x} dx = \int_0^{\pi/2} \frac{\sec^2 x}{1+(1+a)\tan^2 x} dx$
- Putting $t = \tan x$ $\therefore dt = \sec^2 x \; dx$
- When $x = 0, t = 0$; when $x = \pi/2,\; t = \infty$
- $\frac{dI}{da} = \int_0^{\infty} \frac{dt}{1+(1+a)t^2}$
- $\frac{dI}{da} = \frac{1}{a+1} \int_0^{\infty} \frac{dt}{\left[\sqrt{\frac{1}{(a+1)}}\right]^2 + t^2}$
- $= \frac{1}{a+1}\left[\sqrt{a+1} \cdot \tan^{-1}\left(t\sqrt{a+1}\right)\right]_0^{\infty}$ $= \frac{1}{\sqrt{a+1}} \cdot \frac{\pi}{2}$
- Integrating w.r.t. a, we get,
- $I(a) = \frac{\pi}{2} \int \frac{dx}{\sqrt{a+1}} = \pi\sqrt{a+1} + c$
- Putting $a = 0$, we get, $I(0) = \pi + c$
- But $I(0) = \int_0^{\pi/2} \frac{\log(1)}{\sin^2 x} dx = 0$ $\therefore c = -\pi$
  $\therefore I(a) = \pi\sqrt{a+1} - \pi = \pi\left[\sqrt{a+1} - 1\right]$

7) Prove that $\int_0^\infty e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2/4}$, Given: $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Soln :- Let $I(a) = \int_0^\infty e^{-x^2}\cos ax\,dx$ —— (1)

Apply DUIS

$$\frac{dI}{da} = \int_0^\infty e^{-x^2}\left(-\sin ax\,(x)\right)dx$$

$$= -\int_0^\infty e^{-x^2}\cdot x\sin ax\,dx$$

$$= \int_0^\infty \underbrace{(\sin ax)}_{u}\underbrace{\left(e^{-x^2}(-x)\right)}_{v}dx$$

Integrating by parts

$$\frac{dI}{da} = \sin ax\int_0^\infty e^{-x^2}(-x)\,dx - \int_0^\infty a\cos ax\int e^{-x^2}(-x)\,dx$$

$$\left\{\int e^{-x^2}(-x)\,dx = \int e^t\frac{dt}{2} = \frac{1}{2}e^t = \frac{1}{2}e^{-x^2}\right\}$$

put $-x^2 = t$

$-2x\,dx = dt$

$$\therefore \frac{dI}{da} = \left[\sin ax\left(\frac{e^{-x^2}}{2}\right)\right]_0^\infty - \int_0^\infty a\cos ax\left(\frac{e^{-x^2}}{2}\right)dx$$

$$\overline{da} = \Big[\quad \Big(\frac{\quad}{2} \Big) \Big]_0 - \int_0 a\cos \quad \Big(\frac{\quad}{2} \Big) dx$$

$$\frac{dI}{da} = 0 - \frac{a}{2} \int_0^{\infty} e^{-x^2} \cos ax \, dx$$

$$\frac{dI}{da} = -\frac{a}{2} I(a)$$

$$\frac{dI}{I} = -\frac{a}{2} da$$

Integrating both sides

$$\log [I(a)] = -\frac{a^2}{4} + \log C$$

$$\log \left(\frac{I}{C}\right) = -\frac{a^2}{4}$$

$$\frac{I}{C} = e^{-a^2/4} \Rightarrow I(a) = C e^{-a^2/4}$$

put $a = 0$

$$I(0) = C$$

but from (1) $\quad I(0) = \int_0^{\infty} e^{-x^2} dx$

$$= \frac{\sqrt{\pi}}{2} \quad \text{(by given data)}$$

$\therefore \quad c = \frac{\sqrt{\pi}}{2}$

$\therefore \quad I(a) = \frac{\sqrt{\pi}}{2} e^{-a^2/4}$

8) Prove that $\int_0^\infty \frac{tan^{-1}ax}{x(1+x^2)} dx = \frac{\pi}{2}\log(1+a)$

- Let $I(a) = \int_0^\infty \frac{tan^{-1}ax}{x(1+x^2)} dx$
- By the rule of differentiation under integral sign
- $\frac{dI}{da} = \int_0^\infty \frac{\partial f}{\partial a} dx = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{(1+a^2x^2)} dx$
- $= \frac{1}{1-a^2} \int_0^\infty \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$ [By partial differentiation]
- $\frac{dI}{da} = \frac{1}{1-a^2} [\tan^{-1} x - a \tan^{-1} ax]_0^\infty$
- $= \frac{1}{1-a^2} \left[\frac{\pi}{2} - a \cdot \frac{\pi}{2}\right] = \frac{\pi}{2} \cdot \frac{1}{1+a}$
- Integrating both sides w.r.t. a
- $I(a) = \frac{\pi}{2}\log(1+a) + c$
- To find c, we put $a = 0$ $\therefore I(0) = \frac{\pi}{2}\log(1) + c = c$
- But $I(0) = \int_0^\infty \frac{tan^{-1}0}{x(1+x^2)} dx = \int_0^\infty 0\, dx = 0$ $\therefore c = 0$
- $\therefore I(a) = \frac{\pi}{2}\log(1+a)$
- $\therefore \int_0^\infty \frac{tan^{-1}ax}{x(1+x^2)} dx = \frac{\pi}{2}\log(1+a)$

9) Prove that
$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3} tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$

Soln :- we know that $\int_0^x \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a}$

0

By Rule of DUIS, differentiating wrt a

$$\int_0^x \frac{\partial}{\partial a}\left(\frac{1}{x^2+a^2}\right)dx = \frac{d}{da}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$\int_0^x \frac{-2a}{(x^2+a^2)^2}\,dx = \frac{-1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)+\frac{1}{a}\cdot\frac{1}{1+(x^2/a^2)}\cdot\left(\frac{-x}{a^2}\right)$$

$$\int_0^x \frac{-2a}{(x^2+a^2)^2}\,dx = \frac{-1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)-\frac{x}{a}\cdot\frac{1}{a^2+x^2}$$

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)+\frac{x}{2a^2(a^2+x^2)}$$

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10) Prove that $\int_0^\infty \frac{tan^{-1}(x/a)-tan^{-1}(x/b)}{x}dx = \frac{\pi}{2}log\left(\frac{b}{a}\right) where\ a>0, b\geq a$

Soln:- let $I(a) = \int_0^\infty \frac{\tan^{-1}(x/a)-\tan^{-1}(x/b)}{x}\,dx$ —(1)

Apply DUIS, differentiate wrt a

$$\frac{dI}{da} = \int_0^\infty \frac{1}{x}\frac{\partial}{\partial a}\left[\tan^{-1}\left(\frac{x}{a}\right)-\tan^{-1}\left(\frac{x}{b}\right)\right]dx$$

∞ $\left(\frac{-x}{\ }\right)$ dx

$$= \int_0^\infty \frac{1}{x} \cdot \frac{1}{1+\left(\frac{x^2}{a^2}\right)} \cdot \left(\frac{-x}{a^2}\right) dx$$

$$\frac{dI}{da} = \int_0^\infty \frac{-dx}{x^2+a^2} = -\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]_0^\infty$$

$$\frac{dI}{da} = -\left[\frac{1}{a}\frac{\pi}{2} - 0\right] = \frac{-1}{a}\frac{\pi}{2}$$

$$dI = -\frac{\pi}{2}\frac{1}{a}da$$

Integrating both sides

$$I(a) = -\frac{\pi}{2}\log a + c$$

To find c, put $a=b$

$$I(b) = -\frac{\pi}{2}\log b + c$$

but from (1), $I(b) = \int_0^\infty 0 \, dx = 0$

$$\therefore c = \frac{\pi}{2}\log b$$

$$\therefore I(a) = -\frac{\pi}{2}\log a + \frac{\pi}{2}\log b$$

$$I(a) = \frac{\pi}{2}\log\left(\frac{b}{a}\right)$$

11) Prove that $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$

Sol^n

$$\int_0^\infty e^{-ax}\sin bx\,dx = \left\{\frac{e^{-ax}}{a^2+b^2}\left[-a\sin bx - b\cos bx\right]\right\}_0^\infty$$

$$= \frac{1}{a^2+b^2}\left[0 - \left(-a(0) - b(1)\right)\right]$$

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

Apply DUIS, diff. wrt a

$$\int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\sin bx\right)dx = \frac{d}{da}\left[\frac{b}{a^2+b^2}\right]$$

$$\int_0^\infty (-x)\,e^{-ax}\sin bx\,dx = \frac{-b}{(a^2+b^2)^2}\cdot 2a$$

$$\int_0^\infty x\,e^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$$

12) By differentiating $\int_0^\infty \frac{dx}{x^2+a^2} = \frac{\pi}{2a}$ w.r.t a under the integral sign successively,

prove that $\int_0^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{2^{2n+1}.(n!)^2 a^{2n+1}}$

- Consider $\int_0^{\infty} \frac{dx}{(x^2+a^2)} = \frac{\pi}{2a}$
- we apply the rule of DUIS Differentiating both sides w.r.t. a,
- $\int_0^{\infty} \frac{-2adx}{(x^2+a^2)^2} = \frac{-\pi}{2a^2}$
- $\therefore \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2^2 a^3} = \frac{2!\,\pi}{2^3(1!)^2 a^3}$ ....(1)
- Again by the rule of DUIS $\int_0^{\infty} \frac{-2\cdot 2a}{(x^2+a^2)^3} dx = \frac{\pi}{2^2} \cdot \frac{(-3)}{a^4}$
- $\therefore \int_0^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{\pi\cdot 1\cdot 3}{2^4\cdot a^5} = \frac{4!\,\pi}{2^5(2!)^2 a^5}$ ....(2)
- Again by the rule of DUIS $\int_0^{\infty} \frac{-3\cdot 2a}{(x^2+a^2)^4} dx = \frac{\pi\cdot 1\cdot 3\cdot(-5)}{2^4\cdot a^6}$
- $\therefore \int_0^{\infty} \frac{dx}{(x^2+a^2)^4} dx = \frac{\pi\cdot 1\cdot 3\cdot 5}{2^4\cdot 2\cdot 3\cdot a^7} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2^4\cdot(1\cdot 2\cdot 3)\cdot(2\cdot 4\cdot 6)} \cdot \frac{\pi}{a^7} = \frac{6!\,\pi}{2^7(1\cdot 2\cdot 3)^2 a^7} = \frac{6!\,\pi}{2^7(3!)^2 a^7}$ ....(3)
- Now generalize $\int_0^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{2^{2n+1}.(n!)^2 a^{2n+1}}$

13) Evaluate $\int_0^{\pi/2} \frac{dx}{a^2 sin^2 x + b^2 cos^2 x}$ and show that

$$\int_0^{\pi/2} \frac{dx}{(a^2 sin^2 x + b^2 cos^2 x)^2} = \frac{\pi}{4ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Soln :-

$$I = \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$$

Dividing numerator & denominator by $\cos^2 x$

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{a^2 \tan^2 x + b^2}\, dx$$

put $t = \tan x \Rightarrow \sec^2 x\, dx = dt$

$x = 0, \quad t = 0, \quad$ when $x = \frac{\pi}{2}, \quad t = \infty$

$$I = \int_0^\infty \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int_0^\infty \frac{dt}{t^2 + (b/a)^2}$$

$$= \frac{1}{a^2} \cdot \frac{a}{b} \left[\tan^{-1}\left(\frac{a}{b}\right) t \right]_0^\infty = \frac{1}{ab} \left[\frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi}{2ab}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab} \quad \text{--- (1)}$$

Apply DUIS, Differentiate wrt a

$$\int_0^{\pi/2} \frac{-dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} \cdot 2a \sin^2 x = \frac{\pi}{2b} \left(\frac{-1}{a^2} \right)$$

$$\int_0^{\pi/2} \frac{\sin^2 x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4ab} \left(\frac{1}{a^2} \right) \quad \text{--- (2)}$$

Apply DUIS on (1), differentiate wrt b

$\pi/2$

Apply DUIS [illegible] ,

$$\int_0^{\pi/2} \frac{-\,dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}\cdot 2b\cos^2 x = \frac{\pi}{2a}\left(\frac{-1}{b^2}\right)$$

$$\int_0^{\pi/2} \frac{\cos^2 x\,dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{\pi}{4ab}\left(\frac{1}{b^2}\right) \quad - (3)$$

Adding (2) & (3), we get the required answer