BETA FUNCTION

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Definition :- The function of m and n defined by the integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \,, \quad (m,n>0)$$ is called the Beta function

and is denoted by $B(m,n)$

Thus $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of Beta Function :-

(i) $B(m,n) = B(n,m)$

proof :- By definition, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = 1-t \qquad dx = -dt$

$$= \int_1^0 (1-t)^{m-1}\, t^{n-1}\,(-dt)$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1}\,dt$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= B(n,m)$$

(ii) Relation between Beta and Gamma Functions

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

We prove it at a proper place while studying double integration.

we p... double integration.

(iii) $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \cos^{2n-1}\theta \, d\theta$

Proof :- we have

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta \qquad dx = 2\sin\theta\cos\theta\,d\theta$

$x=0, \ \theta=0 \quad , x=1 , \ \theta = \frac{\pi}{2}$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \ 2\sin\theta\cos\theta\,d\theta$$

$$B(m,n) = 2\int \sin^{2m-1}\theta \ \cos^{2n-1}\theta \ d\theta$$

This can be considered as second form of Beta Function.

(iv) $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Soln :- In the above result

put $p = 2m-1 \ , q = 2n-1$

$$2\int^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{\frac{1}{2}\, \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$$

Ex:- $$\int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta = \frac{1}{2} B\left(\frac{3+1}{2}, \frac{4+1}{2}\right)$$

$$= \frac{1}{2} B\left(2, \frac{5}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(2)\, \Gamma\left(\frac{5}{2}\right)}{\Gamma\left(2+\frac{5}{2}\right)} = \frac{\frac{1}{2} (1)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\frac{1}{2} \cdot \frac{\cancel{3}}{2} \cdot \frac{1}{2}\, \cancel{\Gamma\left(\frac{1}{2}\right)}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{\cancel{3}}{2}\right)\left(\frac{1}{2}\right) \cancel{\Gamma\left(\frac{1}{2}\right)}} \qquad \left(\Gamma(n+1) = n\,\Gamma(n)\right)$$

$$\int_0^{\pi/2} \sin^3\theta \cos^4\theta = \frac{2}{35}$$

(v) $\Gamma\frac{1}{2} = \sqrt{\pi}$

Pf:- we know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\frac{1}{2}\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

put $p = q = 0$

$$\int_0^{\pi/2} d\theta = \frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

(vi) $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n)$

Soln:- $I = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

put $x = \frac{t}{1-t}$

or $x = \tan^2\theta$

$dx = 2\tan\theta \sec^2\theta \, d\theta$

when $x = 0$, $\theta = 0$

when $x = 0, \theta = 0$

$x = \infty, \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}}{(1+\tan^2\theta)^{m+n}} \cdot 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2m-1}\theta \sec^{2-2m-2n}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{2m-1} \left(\frac{1}{\cos\theta}\right)^{2-2m-2n} d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$= B(m,n)$ (using second form of Beta function)

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Revision

① Definition :-

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

Properties

① $B(m,n) = B(n,m)$

② Relation between Beta and Gamma Function

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

③ Second Form of Beta Function

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

④ $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

⑤ $$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n)$$

put $x = \dfrac{t}{1-t}$ or $t = \tan^2\theta$

Duplication Formula of Gamma Function :-

$$2^{2m-1} \Gamma m \, \Gamma m+\tfrac{1}{2} = \sqrt{\pi} \, \Gamma 2m$$

Proof :- we have $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

put $m = n$

$$B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta \cos\theta)^{2m-1} d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta$$

put $2\theta = t \quad \therefore d\theta = \frac{dt}{2}$

$$B(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi} (\sin t)^{2m-1} \frac{dt}{2}$$

$$= \frac{1}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} t \, dt$$

$$\left\{ \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(2a-x) = f(x) \right\}$$

$$B(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} t \, dt$$

$$= \frac{2}{2^{2m-1}} \int^{\pi/2} \sin^{2m-1} t \cos^{0} t \, dt$$

$$= \frac{2}{2^{2m-1}} \int_0 \sin^{2m-1} t \cos^0 t \, dt$$

$$B(m,m) = \frac{1}{2^{2m-1}} \beta\left(m, \frac{1}{2}\right)$$

$$\frac{\Gamma m \, \Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}} \frac{\Gamma m \, \Gamma \frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$2^{2m-1} \Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \Gamma \frac{1}{2} \, \Gamma 2m$$

$$2^{2m-1} \Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi} \, \Gamma 2m.$$

Particular cases.

(1) put $m = \frac{1}{4}$

$$2^{-1/2} \Gamma \frac{1}{4} \, \Gamma \frac{3}{4} = \sqrt{\pi} \, \Gamma \frac{1}{2} \Rightarrow \Gamma \frac{1}{4} \, \Gamma \frac{3}{4} = \sqrt{2} \, \pi$$

(2) putting $m = \frac{3}{4}$

$$2^{1/2} \Gamma \frac{3}{4} \, \Gamma \frac{5}{4} = \sqrt{\pi} \, \Gamma \frac{3}{2} \Rightarrow \Gamma \frac{3}{4} \, \Gamma \frac{5}{4} = \frac{\sqrt{\pi}}{\sqrt{2}} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{\pi}{2\sqrt{2}}$$

Type-I :- Evaluate $\int_0^a x^m (a-x)^n \, dx$

method :- put $x = at$

Type-II :- Evaluate $\int_0^1 x^m (1-x^n)^p \, dx$

method :- put $x^n = t$

Type-III :- integral of trigonometric terms

method :- use $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Type-IV :- Integral of typical algebraic term

Method :- use standard trigonometric substitution.

Type-V :- $\int_a^b (x-a)^m (b-x)^n \, dx$

put $(x-a) = (b-a)t$

Type-VI :- $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} \, dx$

put $bx = \frac{at}{1-t}$ or $a+bx = \frac{a}{1-t}$

or $bx = a\tan^2\theta$.

Type-VII :- $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} \, dx$

put $x = \frac{at}{a+b-bt}$

Type-VIII :- $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx = B(m,n)$

put $x = \frac{t}{1-t}$

Ex-1 :- Evaluate $\int_0^1 (1-\sqrt[5]{x})^{3/2}\, dx$

Sol^n :- Let $I = \int_0^1 (1-\sqrt[5]{x})^{3/2}\, dx$

put $\sqrt[5]{x} = t \Rightarrow x = t^5 \Rightarrow dx = 5t^4\, dt$

$$\therefore I = \int_0^1 (1-t)^{3/2}\, 5t^4\, dt$$

$$= 5\int_0^1 t^4 (1-t)^{3/2}\, dt$$

$$= 5\, B\left(5, \frac{5}{2}\right)$$

$$= 5 \cdot \frac{\Gamma 5\, \Gamma \frac{5}{2}}{\Gamma \frac{15}{2}}$$

$$= \frac{5 \cdot 4! \cdot \Gamma 5/2}{\left(\frac{13}{2}\right)\left(\frac{11}{2}\right)\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma \frac{5}{2}}$$

$$\boxed{\Gamma n+1 = n\,\Gamma n}$$

$$I = \frac{256}{3003}$$

Ex-2 Evaluate $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\, dx$. $\int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

$0 \quad \sqrt{3-x} \qquad 0 \quad \sqrt{1-x^{1/4}}$

Soln: Let $I_1 = \int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx$ and $I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$

In I_1, $x = 3t \rightarrow dx = 3\,dt$

$x=0,\ t=0$ when $x=3,\ t=1$

$$I_1 = \int_0^1 \frac{(3t)^{3/2}}{\sqrt{3-3t}} \cdot 3\,dt = \frac{3^{5/2}}{\sqrt{3}} \int_0^1 \frac{t^{3/2}}{(1-t)^{1/2}}\,dt$$

$$= 3^2 \int_0^1 t^{3/2} (1-t)^{-1/2}\,dt$$

$$I_1 = 9\,B\left(\tfrac{5}{2}, \tfrac{1}{2}\right)$$

$$I_2 = \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$$

put $x^{1/4} = t \rightarrow x = t^4 \rightarrow dx = 4t^3\,dt$

$$I_2 = \int_0^1 \frac{4t^3\,dt}{\sqrt{1-t}} = 4\int_0^1 t^3 (1-t)^{-1/2}\,dt$$

$$I_2 = 4B\left(4, \tfrac{1}{2}\right)$$

$$\therefore I = I_1 \times I_2 = 9B\left(\tfrac{5}{2}, \tfrac{1}{2}\right) \cdot 4B\left(4, \tfrac{1}{2}\right)$$

$$= 36\, \frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} \cdot \frac{\Gamma 4\,\Gamma\frac{1}{2}}{\Gamma \frac{9}{2}}$$

$$\frac{\cdots}{\Gamma 3}\cdot\frac{\cdots}{\Gamma\frac{9}{2}}$$

$$= \frac{36\cdot\Gamma\frac{5}{2}\cdot 3!\ \pi}{2!\cdot\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\frac{5}{2}}$$

$$\therefore I = \frac{432\pi}{35}$$

Ex-3 $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\,dy$

Sol^n:- Let $I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx \qquad I_2 = \int_0^{1/2} \sqrt{2y-4y^2}\,dy$

In I_1, put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

$$I_1 = \int_0^1 \sqrt{1-t}\cdot 2t\,dt = 2\int_0^1 t(1-t)^{1/2}\,dt$$

$$= 2B\left(2, \tfrac{3}{2}\right)$$

$$I_2 = \int_0^{1/2} \sqrt{2y-4y^2}\,dy$$

put $2y = t \Rightarrow dy = \frac{dt}{2}$

when $y=0,\ t=0$, when $y=\frac{1}{2},\ t=1$

$$I_2 = \int_0^1 \sqrt{t-t^2} \cdot \frac{dt}{2} = \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt$$

$$I_2 = \frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$\therefore I = I_1 \times I_2 = 2B\left(2,\frac{3}{2}\right) \times \frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma 2\,\Gamma\frac{3}{2}}{\Gamma\frac{7}{2}} \times \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 3}$$

$$\left[\begin{array}{l} \Gamma n+1 = n\Gamma n \\ \Gamma\frac{1}{2} = \sqrt{\pi} \end{array}\right]$$

$$I = \frac{\pi}{30}$$

Ex-4:- Evaluate $\int_0^{\pi/4} \sin^7 2\theta\,d\theta$

Soln:-

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

Let $I = \int_0^{\pi/4} \sin^7 2\theta\,d\theta$

put $2\theta = t \qquad \theta = \frac{t}{2} \qquad d\theta = \frac{dt}{2}$

$\theta = 0,\ t = 0 \ \big/\ \theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$\therefore I = \int_0^{\pi/2} \sin^7 t\,\frac{dt}{2} = \frac{1}{2}\int_0^{\pi/2} \sin^7 t \cos^0 t\,dt$$

$$\therefore I = \int_0^{\pi/2} \sin^7 t \, \frac{dt}{2} = \frac{1}{2}\int_0^{\pi/2} \sin^7 t \cos^0 t \, dt$$

$$= \frac{1}{2} \cdot \frac{1}{2} B\left(\frac{7+1}{2}, \frac{0+1}{2}\right) = \frac{1}{4} B\left(4, \frac{1}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{\Gamma 4 \, \Gamma \frac{1}{2}}{\Gamma \frac{9}{2}}$$

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$= \frac{1}{4} \cdot \frac{3! \, \Gamma \frac{1}{2}}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma \frac{1}{2}}$$

$$\Gamma n = (n-1)\Gamma n-1$$

$$I = \frac{8}{35}$$

Ex-5 :- $\int_0^{\pi} (1-\cos\theta)^3 \, d\theta$

Soln :- Let $I = \int_0^{\pi} \left(2\sin^2 \frac{\theta}{2}\right)^3 d\theta$

$$= 2^3 \int_0^{\pi} \sin^6\left(\frac{\theta}{2}\right) d\theta$$

put $\frac{\theta}{2} = t$ $\quad d\theta = 2dt$

$\theta = 0, t = 0 \;|\; \theta = \pi, t = \pi/2$

$$I = 2^3 \int_0^{\pi/2} \sin^6 t \cdot 2dt$$

$$= 2^4 \int^{\pi/2} \sin^6 t \cos^0 t \, dt$$

0

$$= 2^4 \int_0^{\pi/2} \sin^6 t \cos^0 t \, dt$$

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = 2^4 \cdot \frac{1}{2} B\left(\frac{6+1}{2}, \frac{0+1}{2}\right) = 8 B\left(\frac{7}{2}, \frac{1}{2}\right)$$

$$= 8 \cdot \frac{\Gamma\frac{7}{2}\,\Gamma\frac{1}{2}}{\Gamma 4} = \frac{8 \cdot \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2} \cdot \Gamma\frac{1}{2}}{3!}$$

$$I = \frac{5\pi}{2} \qquad \left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

Ex-6:- $\displaystyle\int_0^{\pi/6} \cos^3 3\theta \sin^2 6\theta \, d\theta$

Sol^n:- Let $I = \displaystyle\int_0^{\pi/6} \cos^3 3\theta \sin^2 6\theta \, d\theta$

put $3\theta = t \qquad d\theta = \frac{dt}{3}$

$\theta = 0, \ t = 0 \quad \Big| \quad \theta = \frac{\pi}{6}, \ t = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \cos^3 t \sin^2 2t \, \frac{dt}{3}$$

$$= \int_0^{\pi/2} \cos^3 t \, (2 \sin t \cos t)^2 \, \frac{dt}{3}$$

$$4 \int^{\pi/2} \ldots 2 \ldots \cos^5 t \, dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t \, \cos^5 t \, dt$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = \frac{4}{3}\cdot\frac{1}{2} B\left(\frac{2+1}{2}, \frac{5+1}{2}\right) = \frac{2}{3} B\left(\frac{3}{2}, 3\right)$$

$$= \frac{2}{3}\,\frac{\Gamma\frac{3}{2}\,\Gamma 3}{\Gamma\frac{9}{2}} = \frac{2}{3}\cdot\frac{\Gamma\frac{3}{2}\,(2!)}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\frac{3}{2}}$$

$$I = \frac{32}{315}$$

Ex-7:- $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Soln:- $I = \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{-\frac{1}{2}}\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) = \frac{1}{2}\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma 1} = \frac{1}{2}\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}$$

$$= \frac{1}{2} \cdot \sqrt{2}\,\pi \qquad \left(\Gamma\frac{3}{4}\,\Gamma\frac{1}{4} = \sqrt{2}\,\pi\right)$$

$$I = \frac{\pi}{\sqrt{2}}$$

Ex-8 :- $\int_0^{\pi} x \sin^5 x \cos^4 x \, dx$

Soln :- $I = \int_0^{\pi} x \sin^5 x \cos^4 x \, dx$

$$\left\{ \because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right\}$$

$$I = \int_0^{\pi} (\pi - x) \sin^5(\pi - x) \cos^4(\pi - x)\, dx$$

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$$= \int_0^{\pi} (\pi - x) \sin^5 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^5 x \cos^4 x \, dx - \int_0^{\pi} x \sin^5 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi} \sin^5 x \cos^4 x \, dx - I$$

$$2I = \pi \int_0^{\pi} \sin^5 x \cos^4 x \, dx$$

$$2I = \pi \int_0 \sin^5 x \cos^4 x \, dx$$

$$\left\{ \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(x) = f(2a-x) \right\}$$

$$f(x) = \sin^5 x \cos^4 x$$

$$f(\pi - x) = \sin^5(\pi - x)\cos^4(\pi - x) = \sin^5 x \cos^4 x = f(x)$$

$$2I = 2\pi \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$$

$$I = \pi \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = \pi \cdot \frac{1}{2} B\left(\frac{5+1}{2}, \frac{4+1}{2}\right) = \frac{\pi}{2} B\left(3, \frac{5}{2}\right)$$

$$I = \frac{\pi}{2} \cdot \frac{\Gamma 3 \, \Gamma\frac{5}{2}}{\Gamma\frac{11}{2}} = \frac{\pi}{2} \cdot \frac{2! \, \Gamma\frac{5}{2}}{\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\Gamma\frac{5}{2}}$$

$$I = \frac{8\pi}{315}$$

Ex-9 :- $\displaystyle\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/3} \, d\theta$

Soln :- Let $\displaystyle I = \int^{\pi/4} (\sin\theta + \cos\theta)^{1/3} \, d\theta$

$-\pi/4$

$$= \int_{-\pi/4}^{\pi/4} 2^{1/6}\left[\sin\theta\cdot\frac{1}{\sqrt{2}} + \cos\theta\cdot\frac{1}{\sqrt{2}}\right]^{1/3} d\theta$$

$$= \int_{-\pi/4}^{\pi/2} 2^{1/6}\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\cdot\sin\frac{\pi}{4}\right]^{1/3} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 2^{1/6}\left[\sin\left(\theta+\frac{\pi}{4}\right)\right]^{1/3} d\theta$$

put $\theta + \frac{\pi}{4} = t \rightarrow d\theta = dt$

$\theta = -\frac{\pi}{4},\ t = 0 \quad \Big|\ \theta = \frac{\pi}{4},\ t = \frac{\pi}{2}$

$$\therefore I = 2^{1/6}\int_0^{\pi/2} (\sin t)^{1/3}\, dt$$

$$= 2^{1/6}\int_0^{\pi/2} \sin^{1/3} t\, \cos^0 t\, dt$$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = 2^{1/6}\cdot\frac{1}{2}B\left(\frac{\frac{1}{3}+1}{2}, \frac{0+1}{2}\right) = 2^{1/6-1}B\left(\frac{2}{3}, \frac{1}{2}\right)$$

$$= 2^{-5/6}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{2}}{\Gamma 7/\ldots} = 2^{-5/6}\cdot\frac{\Gamma\frac{2}{3}\cdot\sqrt{\pi}}{\Gamma\ldots}$$

$$\frac{\quad}{\Gamma 7/6} = \frac{\Gamma 3 \quad}{\Gamma 7/6}$$

Ex10 :- $\displaystyle\int_0^1 x^5 \sqrt{\frac{(1+x^2)}{(1-x^2)}}\, dx$

Soln :- Let $\displaystyle I = \int_0^1 x^5 \sqrt{\frac{1+x^2}{1-x^2}}\, dx = \int_0^1 x^4 \frac{(1+x^2)}{\sqrt{1-x^4}}\, x\,dx$

put $x^2 = \sin\theta \quad \therefore 2x\,dx = \cos\theta\, d\theta$

$x = 0,\ \theta = 0 \quad \Big| \quad x = 1,\ \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \sin^2\theta \, \frac{(1+\sin\theta)}{\sqrt{1-\sin^2\theta}} \cdot \frac{\cos\theta\, d\theta}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,(1+\sin\theta)\, d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \sin^2\theta\, d\theta + \int_0^{\pi/2} \sin^3\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \sin^2\theta \cos^0\theta\, d\theta + \int_0^{\pi/2} \sin^3\theta\cos^0\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}B\left(\frac{2+1}{2}, \frac{0+1}{2}\right) + \frac{1}{2}B\left(\frac{3+1}{2}, \frac{0+1}{2}\right)\right]$$

$$= \frac{1}{4}\left[B\left(\frac{3}{2}, \frac{1}{2}\right) + B\left(2, \frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} + \frac{\Gamma 2\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{2}} \right]$$

$$= \frac{1}{4}\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}\sqrt{\frac{1}{2}}}{(1)} + \frac{(1)\,\Gamma\frac{1}{2}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\Gamma\frac{1}{2}\right)} \right]$$

$$I = \frac{1}{8}\pi + \frac{1}{3}$$

Ex-11 :- $\int_0^1 x^5 \sin^{-1}x \, dx$

LIATE

Sol^n :- Let $I = \int_0^1 x^5 \sin^{-1}x \, dx$

Integrating by parts, we have

$$I = \left[\sin^{-1}x \cdot \frac{x^6}{6}\right]_0^1 - \int_0^1 \left(\frac{x^6}{6}\right)\left(\frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \left[\sin^{-1}(1)\cdot\frac{1}{6}\right] - \frac{1}{6}\int_0^1 \frac{x^6}{\sqrt{1-x^2}}\, dx$$

put $x = \sin t$ $\quad dx = \cos t \, dt$

$x = 0,\ t = 0 \qquad x = 1,\ t = \pi/2$

$$I = \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \frac{\sin^6 t}{\sqrt{1-\sin^2 t}} \cdot \cos t \, dt$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^{\pi/2} \sin^6 t \, dt$$

$$= \frac{\pi}{12} - \frac{1}{6} \int_0^{\pi/2} \sin^6 t \cos^0 t \, dt$$

$$= \frac{\pi}{12} - \frac{1}{6} \cdot \frac{1}{2} B\left(\frac{6+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{\pi}{12} - \frac{1}{12} B\left(\frac{7}{2}, \frac{1}{2}\right)$$

$$= \frac{\pi}{12} - \frac{5\pi}{192}$$

$$I = \frac{11\pi}{192}$$

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Ex-12 :- Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^{9/2}}$

Sol^n :- $I = \int_0^\infty \frac{dx}{(1+x^2)^{9/2}}$

put $x = \tan\theta$ $\quad \therefore dx = \sec^2\theta \, d\theta$

when $x = 0$, $\theta = 0$

when $x = \infty$, $\theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{9/2}}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^9\theta} d\theta = \int_0^{\pi/2} \cos^7\theta \, d\theta$$

$$\int^{\pi/2} \sin^0\theta \cos^7\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^0\theta \cos^7\theta \, d\theta$$

Now $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$I = \frac{1}{2} B\left(\frac{1}{2}, 4\right) = \frac{1}{2} \cdot \frac{\Gamma\frac{1}{2} \; \Gamma 4}{\Gamma 9/2}$$

$$I = \frac{1}{2} \frac{\Gamma\frac{1}{2} \cdot 3!}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}} = \frac{16}{35}$$

Ex-13 $\int_0^\infty \frac{x^2}{(1+x^6)^{7/2}} dx$

Soln :- Let $I = \int_0^\infty \frac{x^2}{(1+x^6)^{7/2}} dx$

put $x^3 = \tan\theta$ $\therefore 3x^2 dx = \sec^2\theta \, d\theta$

when $x = 0$, $\theta = 0$, when $x = \infty$, $\theta = \pi/2$

$$I = \int_0^{\pi/2} \frac{\frac{1}{3}\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{7/2}} = \frac{1}{3}\int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^7\theta \, d\theta}$$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^5\theta \, d\theta = \frac{1}{3} \cdot \frac{1}{2} B\left(\frac{1}{2}, 3\right) =$$

Ex-14 :- $\int_0^\infty \frac{x^3}{(1+x^8)^4} dx$

Soln :- let $I = \int^\infty \frac{x^3}{(1+x^8)^4} dx$

Soln :- Let $I = \int_0^\infty \frac{x^3}{(1+x^8)^4}\, dx$

put $x^4 = \tan\theta \qquad 4x^3 dx = \sec^2\theta\, d\theta$

$$I = \int_0^{\pi/2} \frac{\frac{1}{4}\sec^2\theta\, d\theta}{(1+\tan^2\theta)^4} = \frac{1}{4}\int_0^{\pi/2} \frac{\sec^2\theta}{\sec^8\theta}\, d\theta$$

$$I = \frac{1}{4}\int_0^{\pi/2} \cos^6\theta\, d\theta = \text{- - - - - - - - - -}$$

Ex-15 :- prove that $\int_0^\infty \frac{1}{(x^2+1)^{n+1}}\, dx = \frac{(2n)!}{2^{2n}\cdot(n!)^2}\cdot\frac{\pi}{2}$

Soln :- Let $I = \int_0^\infty \frac{1}{(x^2+1)^{n+1}}\, dx$

put $x = \tan\theta, \qquad dx = \sec^2\theta\, d\theta$

$x = 0, \ \theta = 0 \quad | \quad x = \infty, \ \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{n+1}} = \int_0^{\pi/2} \frac{\sec^2\theta}{(\sec\theta)^{2n+2}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{(\sec\theta)^{2n}}\, d\theta = \int_0^{\pi/2} \cos^{2n}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^0\theta\, \cos^{2n}\theta\, d\theta$$

$$= \frac{1}{2} B\left(\frac{0+1}{2}, \frac{2n+1}{2}\right) = \frac{1}{2} B\left(\frac{1}{2}, \frac{2n+1}{2}\right)$$

$$I = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{2n+1}{2}}{\Gamma\frac{2n+2}{2}} = \frac{1}{2} \cdot \frac{\Gamma\frac{1}{2}\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2} \cdot \underbrace{\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!}}_{(A)}$$

$$\boxed{\Gamma(n+1) = n\,\Gamma n}$$

Now $\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$$

and so on

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) \cdots\cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\frac{1}{2}$$

$$= \frac{(2n-1)(2n-3)\cdots\cdots 5 \cdot 3 \cdot 1}{2^n} \sqrt{\pi}$$

Multiply the numerater and denominator by

$2n\,(2n-2)\,(2n-4)\cdots\cdots 6 \cdot 4 \cdot 2$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)(2n-1)(2n-2)(2n-3)\cdots\cdots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^n (2n)(2n-2)(2n-4)\cdots\cdots 6 \cdot 4 \cdot 2} \cdot \sqrt{\pi}$$

$$= \frac{(2n)!\,\sqrt{\pi}}{\cdots}$$

$$= \frac{(2n)!\sqrt{\pi}}{2^n \cdot 2^n \cdot n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}(n)!}$$

Substituting in (A)

$$I = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!} = \frac{\sqrt{\pi}}{2}\cdot\frac{(2n)!\sqrt{\pi}}{2^{2n}(n!)}\cdot\frac{1}{n!}$$

$$I = \frac{\pi}{2}\cdot\frac{(2n)!}{2^{2n}(n!)^2}$$

EX-16 Prove that $\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,dx = \frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}$

Soln:- Let $I = \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\,dx$

put $x = \sin\theta$ $\therefore dx = \cos\theta\, d\theta$

when $x=0$, $\theta=0$ | $x=1$, $\theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\sin^{2n}\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\, d\theta = \int_0^{\pi/2} \sin^{2n}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{2n}\theta \cdot \cos^0\theta \, d\theta = \frac{1}{2} B\left(\frac{2n+1}{2}, \frac{1}{2}\right)$$

Same as the previous sum.

Ex-17 :- Prove that $\displaystyle\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{\pi}{n} \operatorname{cosec}\left(\frac{\pi}{n}\right)$

Soln :- $\displaystyle I = \int_0^a \frac{dx}{(a^n - x^n)^{1/n}}$

put $x^n = a^n \sin^2\theta$

$x = a \sin^{(2/n)}\theta \qquad \therefore dx = \frac{2a}{n} \sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta \, d\theta$

when $x = 0$, $\theta = 0$ | when $x = a$, $\theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\frac{2a}{n} \sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta \, d\theta}{(a^n - a^n \sin^2\theta)^{1/n}}$$

$$I = \frac{2a}{n} \int_0^{\pi/2} \frac{\sin^{\left(\frac{2}{n}-1\right)}\theta \cos\theta \, d\theta}{a \cos^{2/n}\theta}$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin^{\left(\frac{2}{n}-1\right)}\theta \cos^{\left(1-\frac{2}{n}\right)}\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$I = \frac{2}{n} \cdot \frac{1}{2} B\left(\frac{1}{n}, 1-\frac{1}{n}\right) = \frac{1}{n} \cdot \frac{\Gamma\frac{1}{n}\,\Gamma\left(1-\frac{1}{n}\right)}{\Gamma 1}$$

We know that $\Gamma p \, \Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$

$$\Gamma\frac{1}{n}\,\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}$$

$$\therefore I = \frac{1}{n} \cdot \frac{\pi}{\sin\frac{\pi}{n}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Ex-18 :- Evaluate $\displaystyle\int_7^{11} \sqrt[4]{(x-7)(11-x)}\, dx$

$$\boxed{\text{Type-5} \quad \int_a^b (x-a)^m (b-x)^n \, dx \qquad \text{put } x-a = (b-a)t}$$

Let $\displaystyle I = \int_7^{11} (x-7)^{1/4} (11-x)^{1/4} \, dx$

put $x-7 = (11-7)t = 4t$

$dx = 4\,dt$

$x = 4t+7$

$11-x = 11-4t-7$
$= 4-4t$

when $x=7$, $t=0$ | when $x=11$, $t=1$

$$\therefore I = \int_0^1 (4t)^{1/4} (4-4t)^{1/4} \cdot 4\,dt$$

$$0$$

$$= 8\int_0^1 t^{1/4}(1-t)^{1/4}\,dt$$

$$\int_0^1 x^m(1-x)^n\,dx = B(m+1, n+1)$$

$$\therefore I = 8\,B\left(\frac{1}{4}+1, \frac{1}{4}+1\right) = 8B\left(\frac{5}{4}, \frac{5}{4}\right)$$

$$= 8\cdot\frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}} = \frac{8\cdot\frac{1}{4}\Gamma\frac{1}{4}\cdot\frac{1}{4}\Gamma\frac{1}{4}}{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\Gamma\frac{1}{2}\right)} \qquad \boxed{\Gamma n+1 = n\Gamma n}$$

$$\therefore I = \frac{2}{3}\frac{\left(\Gamma\frac{1}{4}\right)^2}{\sqrt{\pi}}$$

Ex-19 Prove that $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{a^n b^m}B(m,n)$

and hence, evaluate

(i) $\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\,dx$ (ii) $\int_0^\infty \frac{\sqrt{x}}{(1+2x+x^2)}\,dx$ (H.W.)

Sol^n: Let $I = \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$

$$a+bx = a\left(1+\frac{b}{a}x\right)$$

put $\frac{b}{a}x = \tan^2\theta$

$$x = \frac{a}{b}\tan^2\theta \quad \therefore dx = \frac{a}{b}\cdot 2\tan\theta\sec^2\theta\, d\theta$$

when $x=0,\ \theta=0$ | $x=\infty,\ \theta=\pi/2$

$$I = \int_0^{\pi/2} \frac{\left[\frac{a}{b}\tan^2\theta\right]^{m-1}}{a^{m+n}\left[1+\tan^2\theta\right]^{m+n}} \cdot \frac{a}{b}\cdot 2\tan\theta\sec^2\theta\, d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \tan^{2m-1}\theta\,(\sec\theta)^{2-2m-2n}\, d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} \frac{(\sin\theta)^{2m-1}}{(\cos\theta)^{2m-1}}\cdot(\cos\theta)^{2m+2n-2}\, d\theta$$

$$= \frac{2}{a^n b^m}\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta$$

$$\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\, d\theta = \frac{1}{2}B(m,n)$$

$$I = \frac{2}{a^n b^m}\cdot\frac{1}{2}B(m,n) = \frac{1}{a^n b^m}B(m,n)$$

$$\therefore \int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{a^n b^m}B(m,n)$$

(i) $\int_0^\infty \frac{\sqrt{x}}{(4+4x+x^2)}\,dx = \int_0^\infty \frac{x^{1/2}}{(2+x)^2}\,dx$

$a=2,\ b=1,\ m-1=\frac{1}{2},\ m+n=2$

$m=\frac{3}{2},\ n=\frac{1}{2}$

$= \frac{1}{2^{1/2}(1)^{3/2}} B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \cdot \frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2}$

$= \frac{1}{\sqrt{2}} \cdot \frac{\left(\frac{1}{2}\right)\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1!} = \frac{\pi}{2\sqrt{2}}$

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Ex-20 : Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \frac{B(m,n)}{2^m}$

and hence evaluate

(i) $\int_0^1 \frac{x^3-2x^4+x^5}{(1+x)^7}\,dx$ (ii) $\int_0^1 \frac{x-2x^2+x^3}{(1+x)^5}$

Solⁿ :- Using Type-VII of problems

$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx$ put $x=\frac{at}{a+b-bt}$

let $I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx$

comparing $a=1,\ b=1$

put $x=\frac{t}{2-t}$ $\therefore 1-x = 1-\frac{t}{2-t} = \frac{2(1-t)}{2-t}$

$$\cdots \qquad \overline{2-t} \qquad\qquad 2-t \qquad 2-t$$

$$1+x = 1+\frac{t}{2-t} = \frac{2}{2-t} \qquad dx = \frac{(2-t)\cdot 1 - t(-1)}{(2-t)^2}\,dt = \frac{2}{(2-t)^2}\,dt$$

when $x=0$, $t=0$ when $x=1$, $t=1$

$$\therefore I = \int_0^1 \frac{\left(\frac{t}{2-t}\right)^{m-1}\left(\frac{2(1-t)}{2-t}\right)^{n-1}}{\left(\frac{2}{2-t}\right)^{m+n}} \cdot \frac{2}{(2-t)^2}\,dt$$

$$I = \int_0^1 (t)^{m-1}(1-t)^{n-1}\,\frac{2^n}{2^{m+n}} \cdot \frac{(2-t)^{m+n}}{(2-t)^{m+n}}\,dt$$

$$= \frac{1}{2^m}\int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \frac{1}{2^m}B(m,n)$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \frac{B(m,n)}{2^m} \text{ ——— } (1)$$

(i) $$I = \int_0^1 \frac{x^3-2x^4+x^5}{(1+x)^7}\,dx = \int_0^1 \frac{x^3(1-2x+x^2)}{(1+x)^7}\,dx$$

$$= \int_0^1 \frac{x^3(1-x)^2}{(1+x)^7}\,dx$$

comparing with (1) $m=4$, $n=3$

$$= \frac{B(m,n)}{2^m} = \frac{B(4,3)}{2^4} = \frac{\Gamma 4 \, \Gamma 3}{\Gamma 7 \cdot 2^4}$$

$$= \frac{3! \, 2!}{6! \, 2^4} = \frac{1}{960}$$

Ex-22 Prove that $B(n,n) = \frac{1}{2^{2n-1}} B(n, \frac{1}{2})$

Soln :- $B(n,n) = \frac{\Gamma n \, \Gamma n}{\Gamma 2n}$

But Duplication formula gives

$$\Gamma m \, \Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$$

$$\frac{\Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1} \Gamma\left(m+\frac{1}{2}\right)}$$

$$B(n,n) = \frac{\sqrt{\pi}}{2^{2n-1} \Gamma\left(n+\frac{1}{2}\right)} \Gamma n = \frac{1}{2^{2n-1}} \frac{\Gamma \frac{1}{2} \, \Gamma n}{\Gamma\left(n+\frac{1}{2}\right)} = \frac{1}{2^{2n-1}} B\left(n, \tfrac{1}{2}\right)$$

$= RHS.$

Ex 23 :- Prove that $B(m,n) = B(m,n+1) + B(m+1,n)$

Ex 23 :- prove that $B(m,n) = B(m,n+1) + B(m+1,n)$

Soln :- $B(m,n+1) + B(m+1,n)$

$$= \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1} + \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} = \frac{\Gamma m \cdot n \Gamma n + m \Gamma m \cdot \Gamma n}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n \, (m+n)}{(m+n) \, \Gamma m+n} \qquad \left[\because \Gamma n+1 = n \Gamma n \right]$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = B(m,n) = LHS.$$

Ex-24 :- prove that $B(m,m) \cdot B(m+\frac{1}{2}, m+\frac{1}{2}) = \frac{\pi}{m} 2^{1-4m}$

Soln :- LHS :- $B(m,m) \cdot B(m+\frac{1}{2}, m+\frac{1}{2})$

$$= \frac{\Gamma m \, \Gamma m}{\Gamma 2m} \cdot \frac{\Gamma m+\frac{1}{2} \, \Gamma m+\frac{1}{2}}{\Gamma 2m+1} = \frac{\Gamma m \, \Gamma m}{\Gamma 2m} \cdot \frac{\Gamma m+\frac{1}{2} \, \Gamma m+\frac{1}{2}}{2m \, \Gamma 2m}$$

$$= \frac{1}{2m} \cdot \left[\frac{\Gamma m \, \Gamma m+\frac{1}{2}}{\Gamma 2m} \right]^2$$

By Duplication Formula :- $\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$

$$\frac{\Gamma m \, \Gamma m+\frac{1}{2}}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}} \Rightarrow \left[\frac{\Gamma m \, \Gamma m+\frac{1}{2}}{\Gamma 2m} \right]^2 = \frac{\pi}{2^{4m-2}}$$

$$LHS = \frac{1}{2m} \cdot \frac{\pi}{2^{4m-2}} = \frac{\pi}{m} 2^{1-4m} = RHS$$

Ex-25 :- Prove that $B(m+1,n) = \frac{m}{m+n} B(m,n)$

Soln:- $B(m+1,n) = \dfrac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+1+n)} = \dfrac{(m\,\Gamma m)\,\Gamma n}{(m+n)\,\Gamma(m+n)}$

$$= \frac{m}{m+n}\cdot\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \frac{m}{m+n}\,B(m,n)$$

Ex-26:- Prove that $B(n,n+1) = \dfrac{1}{2}\dfrac{(\Gamma n)^2}{\Gamma 2n}$

Hence deduce that $\displaystyle\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta}-\frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta\,d\theta = \frac{(\Gamma\frac{1}{4})^2}{2\sqrt{\pi}}$

Soln:- $B(n,n+1) = \dfrac{\Gamma n\,\Gamma(n+1)}{\Gamma(2n+1)} = \dfrac{\Gamma n\,(n\,\Gamma n)}{2n\,\Gamma 2n} = \dfrac{1}{2}\cdot\dfrac{(\Gamma n)^2}{\Gamma 2n}$

$$I = \int_0^{\pi/2}\left(\frac{1}{\sin^3\theta}-\frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\frac{(1-\sin\theta)^{1/4}}{\sin\theta^{3/4}}\cos\theta\,d\theta$$

put $\sin\theta = t$ $\therefore \cos\theta\,d\theta = dt$

$\theta = 0,\ t=0$ | $\theta = \pi/2\quad t=1$

$$I = \int_0^1\frac{(1-t)^{1/4}}{t^{3/4}}\,dt = \int_0^1 t^{-3/4}(1-t)^{1/4}\,dt$$

$$= B\left(\tfrac{1}{4},\tfrac{5}{4}\right) = B\left(\tfrac{1}{4},\tfrac{1}{4}+1\right) = \frac{1}{2}\frac{(\Gamma\frac{1}{4})^2}{\Gamma\frac{1}{2}} = \frac{1}{2}\cdot\frac{(\Gamma\frac{1}{4})^2}{\sqrt{\pi}}$$

Ex-27:- If $B(n,3) = \dfrac{1}{105}$ and n is a positive integer, find n

Soln :- $B(n,3) = \frac{1}{105}$

$$B(n,3) = \frac{\overline{|n}\,\overline{|3}}{\overline{|n+3}}$$

$$B(n,3) = \frac{\overline{|n}\,\overline{|3}}{(n+2)(n+1)n\overline{|n}} \qquad [\because \overline{|n+1} = n \cdot \overline{|n}]$$

$$= \frac{2!}{(n+2)(n+1)n} \qquad [\because \overline{|n} = (n-1)!]$$

By data this is equal to $1/105$

$$\therefore \frac{2}{(n+2)(n+1)n} = \frac{1}{105}$$

$$\therefore (n+2)(n+1)n = 210 = 7 \cdot 6 \cdot 5$$

$$\therefore n = 5$$

Ex-28 :- Given $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, prove that $\overline{|p}\,\overline{|1-p} = \frac{\pi}{\sin p\pi}$ $(0<p<1)$ Hence evaluate $\int_0^\infty \frac{dy}{1+y^4}$

Soln :- $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$$

put $x = \tan^2\theta \quad \therefore dx = 2\tan\theta \sec^2\theta \, d\theta$

$x = 0, \; \theta = 0, \quad x = \infty, \; \theta = \pi/2$

$$\int_0^{\pi/2} \frac{(\tan\theta)^{2p-2}}{1+\tan^2\theta} \cdot 2\tan\theta\sec^2\theta\,d\theta = \frac{\pi}{\sin p\pi}$$

$$2\int_0^{\pi/2} (\tan\theta)^{2p-1}\,d\theta = \frac{\pi}{\sin p\pi}$$

$$2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{1-2p}\,d\theta = \frac{\pi}{\sin p\pi}$$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$2 \cdot \frac{1}{2} B\left(\frac{2p-1+1}{2}, \frac{1-2p+1}{2}\right) = \frac{\pi}{\sin p\pi}$$

$$B(p, 1-p) = \frac{\pi}{\sin p\pi}$$

$$\frac{\Gamma p \, \Gamma 1-p}{\Gamma p+1-p} = \frac{\pi}{\sin p\pi}$$

$$\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi}.$$

Now $I = \int_0^{\infty} \frac{dy}{1+y^4}$

put $y^4 = x$ $\therefore 4y^3 \, dy = dx$

$y = x^{1/4}$ $dy = \frac{dx}{4y^3} = \frac{1}{4} x^{-3/4} dx$

$$I = \int_0^{\infty} \frac{\frac{1}{4} x^{-3/4} dx}{1+x} = \frac{1}{4} \int_0^{\infty} \frac{x^{-3/4}}{1+x} dx$$

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$$ (Given)

$$I = \frac{1}{4} \cdot \frac{\pi}{\sin\left(\frac{1}{4}\right)\pi}$$ $\left(p-1 = -3/4 \Rightarrow p = 1/4\right)$

$$= \frac{1}{4} \cdot \frac{\pi}{\sin \pi/4} = \frac{\pi}{4 \cdot \frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}}.$$

Prove that $\int_0^\infty \frac{x}{(1+x^4)^{5/4}} dx. \int_0^\infty \frac{1}{\sqrt{1+x^4}} dx = \frac{\pi}{2\sqrt{2}}$

- $I_1 = \int_0^\infty \frac{x}{(1+x^4)^{5/4}} dx$ and $I_2 = \int_0^\infty \frac{1}{\sqrt{1+x^4}} dx$
- Put $x^4 = t$ $\quad \therefore x = t^{1/4}$ $\quad \therefore dx = \frac{1}{4} t^{-3/4} dt$
- When $x = 0, t = 0$; when $x = \infty, t = \infty$
- $\therefore I_1 = \int_0^\infty \frac{1}{(1+t)^{5/4}} \cdot t^{1/4} \cdot \frac{1}{4} \cdot t^{-3/4}\, dt$
- $= \frac{1}{4} \int_0^\infty \frac{t^{-1/2}}{(1+t)^{5/4}} dt = \int_0^\infty \frac{t^{(1/2)-1}}{(1+t)^{(1/2)+(3/4)}} dt \quad = \frac{1}{4} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right)$
  - $\therefore I_2 = \int_0^\infty \frac{1}{(1+t)^{1/2}} \cdot \frac{1}{4} \cdot t^{-3/4}\, dt = \frac{1}{4} \int_0^\infty \frac{t^{(1/4)-1}}{(1+t)^{(1/4)+(1/4)}} dt \quad = \frac{1}{4} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right)$
- $\therefore I = I_1 \times I_2 = \frac{1}{4} \cdot B\left(\frac{1}{2}, \frac{3}{4}\right) \times \frac{1}{4} \cdot B\left(\frac{1}{4}, \frac{1}{4}\right)$
- $\therefore I \quad = \frac{1}{16} \cdot \frac{\left|\overline{1/2}\right. \left|\overline{3/4}\right.}{\left|\overline{5/4}\right.} \cdot \frac{\left|\overline{1/4}\right. \left|\overline{1/4}\right.}{\left|\overline{1/2}\right.} \quad = \frac{1}{16} \cdot \frac{\left|\overline{3/4}\right. \left(\left|\overline{1/4}\right.\right)^2}{(1/4)\left|\overline{1/4}\right.} = \frac{1}{4} \cdot \left|\overline{\tfrac{3}{4}}\right. \left|\overline{\tfrac{1}{4}}\right. \quad = \frac{1}{4} \cdot \sqrt{2} \cdot \pi = \frac{\pi}{2\sqrt{2}}$
-
  - **Alternatively:**
- We may put $x^2 = \tan \theta$.

Show that $\left|\overline{\tfrac{3}{2} - x}\right. \left|\overline{\tfrac{3}{2} + x}\right. = \left(\frac{1}{4} - x^2\right) \pi \sec x\pi, \qquad (-1 < 2x < 1)$

- Since $\left|\overline{n}\right. = (n-1)\left|\overline{n-1}\right.$, we have
- LHS $= \left|\overline{\tfrac{3}{2} - x}\right. \left|\overline{\tfrac{3}{2} + x}\right.$
- $= \left(\frac{1}{2} - x\right) \left|\overline{\tfrac{1}{2} - x}\right. \cdot \left(\frac{1}{2} + x\right) \left|\overline{\tfrac{1}{2} + x}\right. = \left(\frac{1}{4} - x^2\right) \left|\overline{\tfrac{1}{2} - x}\right. \left|\overline{\tfrac{1}{2} + x}\right.$ ………… (1)
  - but $\left|\overline{p}\right. \left|\overline{1 - p}\right. = \frac{\pi}{\sin p\, \pi}$
  - Putting $p = \frac{1}{2} + x$, we get, $\left|\overline{\tfrac{1}{2} + x}\right. \left|\overline{\tfrac{1}{2} - x}\right. = \frac{\pi}{\sin[(1/2)+x]\pi} = \frac{\pi}{\sin\left(\frac{\pi}{2} + \pi x\right)} = \frac{\pi}{\cos \pi x}$
- Hence, from (1), we get,
- LHS $= \left(\frac{1}{4} - x^2\right) \cdot \frac{\pi}{\cos \pi x}$
- $= \left(\frac{1}{4} - x^2\right) \cdot \pi \sec \pi x$

Prove that $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}$ Hence, evaluate **(i)** $\int_0^1 \frac{x^5 + x^8}{(1+x)^{15}} dx$ **(ii)** $\int_0^1 \frac{x^2 + x^3}{(1+x)^7} dx$

- Let $I_1 = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx, \; I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$
- In I_1, put $x = \frac{t}{1-t}$ $\quad \therefore 1 + x = \frac{1}{1-t}$ $\quad \therefore dx = \frac{1}{(1-t)^2} dt$

- When $x = 0, t = 0$; when $x = 1, t = 1/2$
- $\therefore I_1 = \int_0^{1/2} \left(\frac{t}{1-t}\right)^{m-1} \cdot (1-t)^{m+n} \cdot \frac{dt}{(1-t)^2} = \int_0^{1/2} t^{m-1} \cdot (1-t)^{n-1}\, dt$
- Similarly, $I_2 = \int_0^{1/2} t^{n-1} \cdot (1-t)^{m-1}\, dt$
- Now, put $t = 1 - z$ in I_2
- $\therefore\ dt = -dz$ When $t = 0, z = 1$ and when $t = 1/2\,, z = 1/2$
- $\therefore I_2 = \int_1^{1/2} (1-z)^{n-1} z^{m-1} (-dz) = \int_{1/2}^{1} t^{m-1} \cdot (1-t)^{n-1}\, dt$
- $\therefore I \ = I_1 + I_2 = \int_0^{1/2} t^{m-1} \cdot (1-t)^{n-1}\, dt + \int_{1/2}^{1} t^{m-1} \cdot (1-t)^{n-1}\, dt$
- $= \int_0^1 t^{m-1}(1-t)^{n-1}\, dt = B(m,n)$
-
    - Putting the particular values of m, n
- **(i)** $\int_0^1 \frac{x^5+x^8}{(1+x)^{15}} dx = B(6,9)$
- **(ii)** $\int_0^1 \frac{x^2+x^3}{(1+x)^7} dx = B(3,4)$

Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$ & hence evaluate $\int_0^\infty sech^8 x\, dx$

- We have $I = \int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(e^x+e^{-x})^n}$
- Put $e^x = \tan\theta$ $\quad \therefore e^x dx = \sec^2\theta\, d\theta \quad \therefore dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$
- When $x = \infty, e^x = \infty, \tan\theta = \infty \quad \therefore \theta = \pi/2$
- When $x = -\infty, e^x = 0, \tan\theta = 0 \quad \therefore \theta = 0$
- $\therefore I \ = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\tan\theta+\cot\theta)^n} \cdot \frac{\sec^2\theta}{\tan\theta} \cdot d\theta$
- $= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^n} \cdot \frac{1}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot d\theta$
    - $= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^n\theta \cos^n\theta}{\sin\theta\cos\theta} \cdot d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\theta \cos^{n-1}\theta \cdot d\theta$
    - $= \frac{1}{2} \cdot \frac{1}{2} \cdot B\left(\frac{n-1+1}{2}, \frac{n-1+1}{2}\right) = \frac{1}{4} B\left(\frac{n}{2}, \frac{n}{2}\right)$
    - Since, $\frac{e^x+e^{-x}}{2} = \cosh x\,,\quad e^x + e^{-x} = 2\cosh x$
    - Putting $n = 8$ in the integral,
    - $\therefore \int_0^\infty \frac{dx}{(e^x+e^{-x})^8} = \int_0^\infty \frac{dx}{2^8 \cosh^8 x} = \frac{1}{4} B(4,4)$
    - $\therefore \int_0^\infty \text{sech}^8 x\, dx = \frac{2^8}{4} \cdot \frac{\lceil 4 \lceil 4}{\lceil 8} = 2^6 \cdot \frac{3!\cdot 3!}{7!} = \frac{16}{35}$

Prove that $\int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}} dx = \frac{\left(\lceil 3/4\right)^2}{2\sqrt{2\pi}}$

- $I = \int_0^\pi \frac{\sqrt{\sin x}}{(5+3\cos x)^{3/2}} dx$
- Put $t = \tan\frac{x}{2}$, $\quad \sin x = \frac{2t}{(1+t^2)}$, $\quad \cos x = \frac{(1-t^2)}{(1+t^2)}$, $\quad dx = \frac{2dt}{(1+t^2)}$

- When $x = 0, t = 0$; when $x = \pi, t = \infty$
- $\therefore I \quad = \int_0^\infty \frac{\sqrt{2t/(1+t^2)}}{\left[5+3\cdot\left(\frac{1-t^2}{1+t^2}\right)\right]^{3/2}} \cdot \frac{2dt}{(1+t^2)}$
- $= \int\limits_0^\infty \frac{2\sqrt{2}\cdot\sqrt{t}dt}{(8+2t^2)^{3/2}} = \int\limits_0^\infty \frac{\sqrt{t}}{(4+t^2)^{3/2}} dt$
  - Putting $t^2 = 4y, t = 2\sqrt{y}$ $\qquad \therefore dt = \frac{dy}{\sqrt{y}}$
- When $t = 0,\ y = 0$; when $t = \infty, y = \infty$
- $\therefore I \quad = \frac{1}{8}\int_0^\infty \frac{\sqrt{2}\cdot y^{1/4}}{(1+y)^{3/2}} \cdot \frac{dy}{\sqrt{y}} = \frac{1}{4\sqrt{2}}\int_0^\infty \frac{y^{-1/4}}{(1+y)^{3/2}} \cdot dy$
- $= \frac{1}{4\sqrt{2}} B\left(\frac{3}{4}, \frac{3}{4}\right)$
  - $= \frac{1}{4\sqrt{2}} \cdot \frac{\Gamma(3/4)\cdot\Gamma(3/4)}{\Gamma(3/2)} = \frac{1}{4\sqrt{2}} \cdot \frac{\left(\Gamma(3/4)\right)^2}{(1/2)\Gamma(1/2)} = \frac{\left(\Gamma(3/4)\right)^2}{2\sqrt{2\pi}}$