Problems on Series expansion

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Methods of Expansions of functions in power series

1) using Maclaurin's series ✓
2) Using standard expansions ✓
3) Method of Inversion ✗
4) Method of differentiation or Integration of known series ✗
5) Method of Substitution ✗
6) Method using Leibnitz theorem ✗

Ex-1 By Maclaurin's series expand $\log(1+e^x)$ in powers of x upto x^4

Soln:-

$$f(x) = \log(1+e^x) \qquad f(0) = \log(2)$$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)\cdot e^x - e^x \cdot e^x}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2} \qquad f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 \cdot e^x - 2e^x(1+e^x)\cdot e^x}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$f^{(iv)}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x})\cdot 3(1+e^x)^2 \cdot e^x}{(1+e^x)^6}$$

$$= \frac{(1+e^x)(e^x-2e^{2x})-(e^x-e^{2x})\cdot 3e^x}{(1+e^x)^4}$$

$f(0)=\log 2$, $f'(0)=\frac{1}{2}$, $f''(0)=\frac{1}{4}$, $f'''(0)=0$, $f^{(iv)}(0)=-\frac{1}{8}$

Using Maclaurin's series

$$f(x)= f(0)+ x f'(0)+ \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0)+ \frac{x^4}{4!} f^{(iv)}(0) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$$

Ex:- Prove that, $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \cdots$

Soln:- Let $y = \log \sec x$ $\qquad y(0) = \log \sec 0 = \log 1 = 0$

$y_1 = \frac{1}{\sec x}\cdot \sec x \tan x = \tan x$ $\qquad y_1(0) = \tan 0 = 0$

$y_2 = \sec^2 x = 1+\tan^2 x = 1+y_1^2$ $\qquad y_2(0) = 1+y_1(0)^2 = 1$

$y_3 = 2y_1y_2$ $\qquad y_3(0) = 2y_1(0)y_2(0) = 0$

$y_4 = 2[y_2y_2 + y_1y_3]$ $\qquad \therefore y_4(0) = 2[y_2(0)^2 + y_1(0)y_3(0)]$

$= 2y_2^2 + 2y_1y_3$ $\qquad y_4(0) = 2$

$y_5 = 4y_2\cdot y_3 + 2y_2y_3 + 2y_1y_4$ $\qquad y_5(0) = 0$

$= 6y_2y_3 + 2y_1y_4$

$y_6 = 6y_3y_3 + 6y_2y_4 + 2y_2y_4 + 2y_1y_5$ $\qquad y_6(0) = 6(0) + 8(1)(2)$

$= 6y_3^2 + 8y_2y_4 + 2y_1y_5$

$+2(0)$

$y_6(0) = 16$

Hence by Maclaurin's series

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0)$$

$$+ \frac{x^5}{5!} y_5(0) + \frac{x^6}{6!} y_6(0) + \cdots$$

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{x^6}{45} + \cdots$$

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Ex-3 :- Prove that, $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

solution :- $f(x) = \log(1+\sin x)$ $\quad f(0) = \log(1) = 0$

$f'(x) = \frac{\cos x}{1+\sin x}$ $\quad f'(0) = 1$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

$$f'''(x) = \frac{1}{(1+\sin x)^2} \cdot \cos x$$

$\therefore f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 1$

Using Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Aliter :- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\log(1+\sin x) = \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + \cdots$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^2$$

$$+ \frac{1}{3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3 - \cdots$$

$$= x - \frac{1}{2}x^2 + \left(-\frac{x^3}{6} + \frac{x^3}{3}\right) + \cdots$$

$$\log(1+\sin x) = x - \frac{1}{2}x^2 + \frac{x^3}{6} + \cdots$$

Ex-4 Prove that $\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \cdots$

Soln :- $f(x) = \sec^2 x =$ $\qquad f(0) = \sec^2 0 = 1$

$f'(x) = 2\sec^2 x \tan x =$ $\qquad f'(0) = 0$

$f''(x) = 2\left[\sec^4 x + 2\sec^2 x \tan^2 x\right]$ $\qquad f''(0) = 2$

$= 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f'''(x) = 8\sec^4 x \tan x + 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x$

$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$ $\qquad f'''(0) = 0$

$$f^{iv}(x) = 64\sec^4 x\tan^2 x + 16\sec^6 x + 16\sec^2 x\tan^4 x + 24\sec^4 x\tan^2 x \qquad f^{(iv)}(0) = 16$$

$f(0)=1,\ f'(0)=0,\ f''(0)=2,\ f'''(0)=0,\ f^{(iv)}(0)=16$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots$$

$$\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \cdots$$

Method of standard expansion

Ex-5 :- Expand in powers of x, $e^{x\sin x}$

Soln :- let $x\sin x = y$

$$e^{x\sin x} = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

$$e^{x\sin x} = 1 + (x\sin x) + \frac{1}{2!}(x\sin x)^2 + \frac{1}{3!}(x\sin x)^3 + \cdots$$

$$= 1 + x\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] + \frac{1}{2!}x^2\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]^2 + \frac{1}{3!}x^3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]^3 + \cdots$$

$$\frac{\ }{3!}x\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right]+\cdots$$

$$=1+x^2+x^4\left(-\frac{1}{3!}+\frac{1}{2!}\right)+x^6\left(\frac{1}{5!}-\frac{2}{2!3!}+\frac{1}{3!}\right)+\cdots$$

$$e^{x\sin x}=1+x^2+\frac{1}{3}x^4+\frac{1}{120}x^6+\cdots$$

Ex-6 Expand $\log(1+x+x^2+x^3)$ upto x^8

Soln:- $\log(1+x+x^2+x^3)=\log\left[(1+x)(1+x^2)\right]$

$$=\log(1+x)+\log(1+x^2)$$

We know that $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

$$\log(1+x+x^2+x^3)=\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\cdots\right]$$

$$+\left[x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\cdots\right]$$

$$=x+\frac{x^2}{2}+\frac{x^3}{3}-\frac{3}{4}x^4+\frac{x^5}{5}+\frac{1}{6}x^6+\frac{x^7}{7}-\frac{3}{8}x^8+\cdots$$

Ex :- Prove that $x\operatorname{cosec}x=1+\frac{x^2}{6}+\frac{7x^4}{360}+\cdots$

Soln:- $x\operatorname{cosec}x=\frac{x}{\sin x}=\frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots}$

$\sin x \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$= \frac{1}{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots} = \frac{1}{1 - \left[\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right]}$$

We know that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$$x \operatorname{cosec} x = 1 + \left[\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right] + \left[\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right]^2 + \cdots\cdots$$

$$= 1 + \left[\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right] + \left[\frac{x^4}{36} - \frac{2x^6}{6 \times 120} + \frac{x^8}{(120)^2} + \cdots\right]$$

$$= 1 + \frac{x^2}{6} + x^4\left(-\frac{1}{120} + \frac{1}{36}\right) + \cdots\cdots$$

$$x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \cdots\cdots$$

Ex:- Show that $\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots\cdots$

Soln:- $\sin(e^x - 1) = \sin\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right) - 1\right]$

$$= \sin\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\right)$$

But $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\cdots$

$$\sin(e^x-1) = \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) - \frac{1}{3!}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3 + \cdots$$

$$= \left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right) - \frac{1}{6}\left(x^3+\frac{3x^4}{2!}+3\frac{x^5}{4}\cdots\right)+\cdots$$

$$= x + \frac{x^2}{2!} + x^3\left(\frac{1}{6}-\frac{1}{6}\right) + x^4\left(\frac{1}{4!}-\frac{3}{6\times 2!}\right)+\cdots$$

$$\sin(e^x-1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \cdots$$

Ex :- $e^{e^x} = e\left[1+x+x^2+\frac{5}{6}x^3+\frac{5}{8}x^4+\cdots\right]$

Sol^n :- we know $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

$$e^x = 1+y \quad \text{where } y = x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

Now $e^{e^x} = e^{1+y} = e.e^y$

$$= e\left[1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\cdots\right]$$

$$\therefore e^{e^x} = e\left[1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)^2\right.$$

$$\left.+\frac{1}{3!}\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3+\frac{1}{4!}\left(x+\frac{x^2}{2!}+\cdots\right)^4+\cdots\right]$$

$$= e\left[1+\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+\frac{1}{2!}\left(x^2+x^3+\frac{x^4}{4}+\frac{2x^4}{3!}+\cdots\right)\right.$$

$$\left.+\frac{1}{3!}\left(x^3+\frac{3x^4}{2!}+\cdots\right)+\frac{1}{4!}\left(x^4\right)\right]$$

$$= e\left[1+x+x^2\left(\frac{1}{2!}+\frac{1}{2!}\right)+x^3\left(\frac{1}{3!}+\frac{1}{2!}+\frac{1}{3!}\right)+x^4\left(\frac{1}{4!}+\frac{1}{8}+\frac{1}{3!}\right.\right.$$

$$\left.\left.+\frac{1}{4}+\frac{1}{4!}\right)+\cdots\right]$$

$$= e\left[1+x+x^2+\frac{5}{6}x^3+\frac{5}{8}x^4+\cdots\right]$$

Ex:- prove that $(1+x)^x = 1+x^2-\frac{x^3}{2}+\frac{5x^4}{6}-\cdots$

Soln:- Let $y=(1+x)^x$

$$\log y = x\log(1+x)$$

$$= x\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right]$$

$$\log y = x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots = z \text{ (say)}$$

$$\therefore y = e^z$$

$$= 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots$$

$$= 1+\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}\cdots\right]+\frac{1}{2!}\left[x^2-\frac{x^3}{2}+\cdots\right]^2$$

$$= 1 + \left[x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \cdots\right] + \frac{1}{2!}\left[x^4 - x^5 + \cdots\right]$$

$$= 1 + x^2 - \frac{x^3}{2} + x^4\left(\frac{1}{3} + \frac{1}{2}\right) + \cdots$$

$$(1+x)^x = 1 + x^2 - \frac{x^3}{2} + \frac{5}{6}x^4 \cdots\cdots$$