Maclaurin's Series

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we know that using Taylor's series expansion, we can write

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots + \frac{x^n}{n!} f^{(n)}(h) + \cdots \infty$$

If we put $h = 0$ in the above expansion, we will get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \infty$$

This series is known as Maclaurin's Series

Using Maclaurin's series, we can find expansion of many standard functions.

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① Expansion of $\sin x$

let $f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(iv)}(x) = \sin x$	$f^{(iv)}(0) = 0$
$f^{v}(x) = \cos x$	$f^{v}(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \cdots$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

Similarly $\cos x = 1 - x^2 + x^4 - x^6$

Similarly $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

H.W $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

Expansion of e^x

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, $f'''(x) = e^x$

$f(0) = 1$, $f'(0) = 1$, $f''(0) = 1$, $f'''(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Similarly replace x by $-x$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

also, $a^x = e^{x(\log a)}$

$$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \cdots$$

* Expansion of $\sinh x$

$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0$

$\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$

let $f(x) = \sinh x$, $f(0) = 0$

$f'(x) = \cosh x$ $f'(0) = 1$

$f''(x) = \sinh x$ $f''(0) = 0$

$f'''(x) = \cosh x \qquad f'''(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 0 + x + \frac{x^2}{2!}(0) + \frac{x^3}{3!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Similary $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

H.W $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

Expansion of $\log(1+x)$

Let $f(x) = \log(1+x) \qquad f(0) = \log(1) = 0$

$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$

$f^{(iv)}(x) = \frac{-6}{(1+x)^4} \qquad f^{(iv)}(0) = -6$

$f^{(v)}(x) = \frac{24}{(1+x)^5} \qquad f^{(v)}(0) = 24$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots$$

$$= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots$$

Replacing x by $-x$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots$$

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* Expansion of $\tanh^{-1} x$

$$\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right]$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \cdots$$

$$\cos^{-1} x = \frac{\pi}{2} - \left[x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \cdots\right]$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} - \cdots$$

Expansion of $(1+x)^m$

$f(x) = (1+x)^m \qquad\qquad f(0) = 1$

$f'(x) = m(1+x)^{m-1} \qquad\qquad f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2} \qquad\qquad f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \qquad\qquad f'''(0) = m(m-1)(m-2)$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \cdots$$

If m is positive integer, we get finite number of term in the rhs of above series.

If $m = -1$, we get from the above expansion

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

If we change x to $-x$ in above expansion

$$(1-x)^{-1} = \frac{1}{(1-x)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$