Taylor's Theorem and Problems

Sunday, March 7, 2021 12:57 PM

Taylor's series

1) Assuming that $f(x+h)$ can be expanded in ascending powers of h, It is expressed as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots \quad (1)$$

The above series is known as Taylor's series.

2) Interchanging x and h, we can express $f(x+h)$ is ascending powers of x,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots \infty$$

3) Replace x by a, h as $(x-a)$ in (1), we get $f(x)$ as a power series in $(x-a)$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

Ex 1:- Apply Taylor's Theorem to find approximate value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3x^2 + 15x - 10$

Soln:- By Taylor's Theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

put $x = 1$ and $h = 0.1$

$f(x) = x^3 + 3x^2 + 15x - 10$ $\qquad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15$ $\qquad f'(1) = 24$

$f''(x) = 6x + 6$ $\qquad f''(1) = 12$

$f'''(x) = 6$ $\qquad f'''(1) = 6$

$$f\left(\frac{11}{10}\right) = f(1) + (0.1)f'(1) + \frac{(0.1)^2}{2!}f''(1) + \frac{(0.1)^3}{3!}f'''(1)$$

$$= 9 + (0.1)(24) + \frac{(0.1)^2}{2}(12) + \frac{(0.1)^3}{6}(6)$$

$$f\left(\frac{11}{10}\right) = 11.46$$

Ex:- Expand e^x in powers of $(x-1)$

Soln:- $\left\{ \text{we know that } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right]$

we will use the formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

we take $a=1$, $f(x) = e^x$

$$e^x = e^1 + (x-1)e^1 + \frac{(x-1)^2}{2!}e^1 + \frac{(x-1)^3}{3!}e^1 + \cdots$$

$$= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$

$$e^x = e\sum_{n=0}^{\infty}\frac{(x-1)^n}{n!}$$

Ex-3 Expand $\tan^{-1}x$ in powers of $\left(x-\frac{\pi}{4}\right)$

Soln:- Let $f(x) = \tan^{-1}x$, $a = \frac{\pi}{4}$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots\cdots$$

$f(x) = \tan^{-1}x$ $\qquad$ $f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \frac{1}{1+x^2}$ $\qquad$ $f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\frac{\pi}{4}\right)^2}$

$f''(x) = \frac{-1}{(1+x^2)^2}\cdot 2x$ $\qquad$ $f''\left(\frac{\pi}{4}\right) = \frac{-2(\pi/4)}{\left[1+\left(\frac{\pi}{4}\right)^2\right]^2}$

$$\tan^{-1}x = \tan^{-1}\frac{\pi}{4} + \left(x-\frac{\pi}{4}\right)\cdot\frac{1}{1+\left(\frac{\pi}{4}\right)^2} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\cdot\frac{\left[-2(\pi/4)\right]}{\left[1+\left(\frac{\pi}{4}\right)^2\right]^2} + \cdots$$

$$= 1 + \left(x-\frac{\pi}{4}\right)\cdot\frac{16}{16+\pi^2} - \left(x-\frac{\pi}{4}\right)^2\cdot\frac{64\pi}{(16+\pi^2)^2} + \cdots\cdots$$

Ex Expand $\log x$ in powers of $(x-1)$. Hence show that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$$

Soln:- $f(x) = \log x$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$f(x) = \log x$, $\quad a = 1 \quad f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$

$f^{iv}(x) = \frac{-6}{x^4} \qquad f^{iv}(1) = -6$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$$= f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2}f''(1) + \frac{(x-1)^3}{6}f'''(1) + \frac{(x-1)^4}{24}f^{(iv)}(1) + \cdots$$

$$= 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

To get $\log 2$, put $x = 2$

$$\log 2 = (1) - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \cdots$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$$

Ex :- Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1.003)$ upto 5 places of decimals. Given $\pi = 3.141593$

Soln :- Let $f(x+h) = \tan^{-1}(x+h)$, $f(x) = \tan^{-1}x$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \quad \text{—(1)}$$

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = \frac{-2x}{(1+x^2)^2}, \quad f'''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$$

$$\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1+x^2} + \frac{h^2}{2} \cdot \left[\frac{-2x}{(1+x^2)^2}\right] + \frac{h^3}{6}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right] + \cdots$$

put $x=1$, $h=0.003$

$$\tan^{-1}(1.003) = \tan^{-1}(1) + \frac{0.003}{1+1^2} + \frac{(0.003)^2}{2}\left[\frac{-2}{2^2}\right] + \frac{(0.003)^3}{6}\left[\frac{4}{2^3}\right]$$

$$\tan^{-1}(1.003) = 0.78690$$

Ex :- By using Taylor's Theorem arrange in powers of x, $7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$

Soln :- We know that

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \cdots$$

Taking $h=2$

Taking $h=2$

$$f(x+2) = f(2) + x f'(2) + \frac{x^2}{2!} f''(2) + \text{------} \qquad (1)$$

$$f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$$

$f(x) = 7 + x + 3x^3 + x^4 - x^5$ $\qquad \therefore f(2) = 17$

$f'(x) = 1 + 9x^2 + 4x^3 - 5x^4$ $\qquad f'(2) = -11$

$f''(x) = 18x + 12x^2 - 20x^3$ $\qquad f''(2) = -76$

$f'''(x) = 18 + 24x - 60x^2$ $\qquad f'''(2) = -174$

$f^{(iv)}(x) = 24 - 120x$ $\qquad f^{(iv)}(2) = -216$

$f^{(v)}(x) = -120$ $\qquad f^{(v)}(2) = -120$

Substituting in (1)

$$f(x+2) = 17 - 11x - 76\frac{x^2}{2!} - 174\cdot\frac{x^3}{3!} - 216\frac{x^4}{4!} - 120\frac{x^5}{5!}$$

$$f(x+2) = 17 - 11x - 38x^2 - 29x^3 - 9x^4 - x^5$$

H.W Find the expansion of $\tan\left(x+\frac{\pi}{4}\right)$ in ascending powers of x upto terms in x^4 and find approximately the value of $\tan(43^\circ)$

Soln: Take $f(x) = \tan x$, $h = \frac{\pi}{4}$

: take $f(x) = \sin x$, $h = \frac{\pi}{4}$

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{(iv)}(h) + \cdots$$

$$43^0 = \frac{\pi}{4} - 2^0 = \left(\frac{\pi}{4} - \frac{2\pi}{180}\right)$$