Leibnitz's Theorem

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STATEMENT

If $y = uv$ where u and v are functions of x possessing derivatives of n^{th} order, then,

$$y_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots\ldots + {}^nC_r u_{n-r} v_r + \ldots\ldots\ldots + {}^nC_n u\, v_n$$

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \cdots + {}^nC_r x^{n-r} y^r + \cdots + {}^nC_n x^0 y^n$$

Given the function, determine the function whose n^{th} derivative is known and treat it as u.

Choose v as that function whose derivative will become zero after certain terms

The Theorem can also be stated as

$$y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \cdots + u v_n$$

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1. Find the nth derivative of $y = x^2 e^{mx}$

Soln :- $y = x^2 e^{mx}$

let $u = e^{mx}$ $\quad\quad v = x^2$

$u_n = m^n \cdot e^{mx}$ $\quad\quad v_1 = 2x$

$u_{n-1} = m^{n-1} e^{mx}$ $\quad\quad v_2 = 2$

$u_{n-2} = m^{n-2} e^{mx}$ $\quad\quad v_3 = 0$

By Leibnitz's Theorem

$$y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \cdots + u v_n$$

$\therefore y_n = m^n e^{mx} x^2 + n.m^{n-1} e^{mx} 2x + \frac{n(n-1)}{2!} m^{n-2} e^{mx}.2 + 0$

$y_n = m^n e^{mx} x^2 + 2n\, m^{n-1} e^{mx} x + n(n-1) m^{n-2} e^{mx}$

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2. find y_n if $y = x^2 \sin x$

Soln :- $u = \sin x$ $\quad V = x^2$

$u_n = \sin\left(x + n\frac{\pi}{2}\right)$ $\quad V_1 = 2x$

$V_2 = 2$

$V_3 = 0$

By Leibnitz theorem

$y_n = u_n v + n u_{n-1} V_1 + \frac{n(n-1)}{2!} u_{n-2} V_2 + \cdots + u V_n$

$y_n = \sin\left(x + n\frac{\pi}{2}\right) \cdot x^2 + n \sin\left(x + (n-1)\frac{\pi}{2}\right) \cdot 2x$
$+ \frac{n(n-1)}{2!} \sin\left(x + (n-2)\frac{\pi}{2}\right) 2 + 0$

$y_n = x^2 \sin\left(x + n\frac{\pi}{2}\right) + 2nx \sin\left(x + (n-1)\frac{\pi}{2}\right)$
$+ n(n-1) \sin\left(x + (n-2)\frac{\pi}{2}\right)$

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3. $y = x \log(x+1)$, prove that $y_n = \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}$

Soln:-

$$y = x\log(x+1)$$

$$y_1 = \log(x+1) + \frac{x}{x+1} = \log(x+1) + 1 - \frac{1}{x+1}$$

diff. (n-1) times $y_n = ?$

let $u = \log(x+1)$, $v = x$

$$u_n = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} , \quad v_1 = 1$$

$$v_2 = 0$$

by Leibnitz theorem

$$y_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \cdots + u v_n$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \cdot x + n \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}(1) + 0$$

$$= \frac{(-1)^{n-2}(-1)(n-1)(n-2)!\, x}{(x+1)^n} + \frac{n(-1)^{n-2}(n-2)!\,(x+1)}{(x+1)^n}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[(-1)(n-1)x + n(x+1)\right]$$

$$y_n = \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}$$

$(x+1)x$

④ If $y = x^n \log x$ prove that $y_{n+1} = \frac{n!}{x}$

Soln :- $y = x^n \log x$

Differentiating wrt x

$$y_1 = n x^{n-1} \log x + x^n \cdot \frac{1}{x}$$

$$= n x^{n-1} \log x + x^{n-1}$$

$$x y_1 = n x^n \log x + x^n$$

$$x y_1 = n y + x^n$$

Apply Leibnitz theorem

$$x y_{n+1} + n(1) y_n = n y_n + n!$$

$$x y_{n+1} + n y_n = n y_n + n!$$

$$x y_{n+1} = n!$$

$$y_{n+1} = \frac{n!}{x}$$

for $x y_1$

$u = y_1 \;, v = x$

$u_n v + n u_{n-1} v_1$

$y_{n+1} x + n y_n (1)$

$x y_{n+1} + n y_n$

$t = x^3$

3 times

$t_1 = 3x^2$

$t_2 = 6x$

$t_3 = 6 = 3!$

⑤ If $y = \cos^{-1} x$ prove that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

Soln :- $y = \cos^{-1} x$

differentiating wrt x

$$y_1 = \frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\; y_1 = -1$$

differentiating wrt x

$$\sqrt{1-x^2}\,(y_2) + y_1\left[\frac{1}{2\sqrt{1-x^2}}\right]\cdot(-2x) = 0$$

multiplying by $\sqrt{1-x^2}$

$$(1-x^2)\,y_2 - x\,y_1 = 0$$

Differentiate n times using Leibnitz theorem

$$\left[y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!}\,y_n(-2)\right]$$

$$-\left[y_{n+1}(x) + n\,y_n(1)\right] = 0$$

$$(1-x^2)\,y_{n+2} - 2nx\,y_{n+1} - n(n-1)\,y_n - x\,y_{n+1} - n\,y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2nx+x)\,y_{n+1} - (n(n-1)+n)\,y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$$

Ex:- If $y = \left(x+\sqrt{a^2+x^2}\right)^m$, prove that at $x=0$

$$a^2 y_{n+2} + (n^2-m^2)\,y_n = 0$$

Soln:- $y = \left(x+\sqrt{a^2+x^2}\right)^m$

Differentiating wrt x

$$y_1 = m\left(x+\sqrt{a^2+x^2}\right)^{m-1}\left[1 + \frac{1}{2\sqrt{a^2+x^2}}\cdot 2x\right]$$

$$= m\left(x+\sqrt{a^2+x^2}\right)^{m-1}\left[\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\right]$$

$$y_1 = \frac{m\left(x+\sqrt{a^2+x^2}\right)^m}{\sqrt{a^2+x^2}}$$

$$y_1 = \frac{my}{\sqrt{a^2+x^2}}$$

$$\sqrt{a^2+x^2}\, y_1 = my \quad \text{——— (1)}$$

differentiating wrt x

$$\sqrt{a^2+x^2}\cdot y_2 + \frac{1}{2\sqrt{a^2+x^2}}\cdot 2x \cdot y_1 = my_1$$

multiply by $\sqrt{a^2+x^2}$

$$(a^2+x^2)\,y_2 + x\,y_1 = m\,y_1\sqrt{a^2+x^2}$$

using (1)

$$(a^2+x^2)y_2 + xy_1 = m^2 y \quad \text{——— (2)}$$

Differentiating each term n times using Leibnitz thm

$$\left[y_{n+2}(a^2+x^2) + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n(2)\right] + \left[y_{n+1}(x) + ny_n(1)\right] = m^2 y_n$$

$$(a^2+x^2)\,y_{n+2} + 2nx\,y_{n+1} + (n^2-n)\,y_n + x\,y_{n+1} + ny_n - m^2y_n = 0$$

$$(a^2+x^2)\,y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)\,y_n = 0$$

put $x = 0$

$$a^2 y_{n+2} + (n^2 - m^2) y_n = 0$$ which is the required answer.

H.W If $y = (x^2-1)^n$ prove that

$$(x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0$$

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Ex-8 If $y = \sin(m \sin^{-1} x)$ or if $m \sin^{-1} x = \sin^{-1} y$, prove that

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$$

Hence deduce that $y_n(0) = 0$ if n is even

and $y_n(0) = ((n-2)^2 - m^2) \ldots\ldots (3^2 - m^2)(1^2 - m^2) m$ if n is odd

Soln :- Given $y = \sin(m \sin^{-1} x)$

Differentiating w.r.t x

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{--- (1)}$$

$$\sqrt{1-x^2}\, y_1 = m \cos(m \sin^{-1} x)$$

differentiating again wrt x

$$\sqrt{1-x^2} \cdot y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x) y_1 = m\left[-\sin(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}}\right]$$

multiply by $\sqrt{1-x^2}$ throughout

$$(1-x^2) y_2 - x y_1 = -m^2 \sin(m \sin^{-1} x)$$

$$(1-x^2) y_2 - x y_1 + m^2 y = 0 \quad \text{--- (2)}$$

Differentiating wrt x by leibnitz rule

$$\left[(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2}(-2)\,y_n\right]$$

$$-\left[x\,y_{n+1} + n(1)\,y_n\right] + m^2 y_n = 0$$

$$(1-x^2)\,y_{n+2} - 2nx\,y_{n+1} - n(n-1)y_n - x\,y_{n+1} - n y_n + m^2 y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)\,y_n = 0 \quad \text{---} \quad (3)$$

Now when $x=0$, $y(0)=0$, $y_1(0)=m$, $y_2(0)=0$

(using (1) & (2))

using (3), put $x=0$

$$y_{n+2}(0) + (m^2-n^2)\,y_n(0) = 0$$

$$y_{n+2}(0) = -(m^2-n^2)\,y_n(0) = (n^2-m^2)\,y_n(0)$$

for n even, ie $n=2 \rightarrow y_4(0) = (2^2-m^2)\,y_2(0) = 0$

$n=4, \rightarrow y_6(0) = (4^2-m^2)\,y_4(0) = 0$

$= (4^2-m^2)(2^2-m^2)\,y_2(0) = 0$

for even n, $y_n(0) = ((n-2)^2-m^2)\cdots(4^2-m^2)(2^2-m^2)\,y_2(0) = 0$

for odd n, ie $n=1$, $y_3(0) = (1^2-m^2)\,y_1(0) = (1^2-m^2)m$

$n=3$, $y_5(0) = (3^2-m^2)\,y_3(0) = (3^2-m^2)(1^2-m^2)\,y_1(0)$

$= (3^2-m^2)(1^2-m^2)m$

$n=5$, $y_7(0) =$ $= (5^2-m^2)(3^2-m^2)(1^2-m^2)m$

for odd n, $y_n(0) = ((n-2)^2-m^2)\cdots\cdots(3^2-m^2)(1^2-m^2)m$

Ex-9 If $y = e^{m\cos^{-1}x}$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Soln :-

$$y = e^{m\cos^{-1}x}$$

differentiating wrt x

$$y_1 = e^{m\cos^{-1}x} \times \frac{-m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = -me^{m\cos^{-1}x}$$

$$\sqrt{1-x^2}\,y_1 = -my$$

differentiating wrt x

$$\sqrt{1-x^2}\,y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x)\,y_1 = -my_1$$

$$(1-x^2)y_2 - xy_1 = -m\sqrt{1-x^2}\,y_1$$

$$= -m(-my)$$

$$(1-x^2)y_2 - xy_1 = m^2y \quad \text{——— (1)}$$

by Leibnitz Thm

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n\right]$$
$$- \left[xy_{n+1} + n(1)y_n\right] = m^2y_n$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Ex-10 If $\sin^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$

prove that, $x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0$

Soln :-

$$\sin^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$$

$$\frac{y}{b} = \sin\left[\log\left(\frac{x}{n}\right)^n\right] = \sin\left[n \log\left(\frac{x}{n}\right)\right]$$

$$y = b \sin\left[n \log\left(\frac{x}{n}\right)\right]$$

differentiating wrt x

$$y_1 = b\cos\left[n\log\left(\frac{x}{n}\right)\right] \cdot n \cdot \frac{n}{x} \cdot \frac{1}{n}$$

$$y_1 = b\cos\left[n\log\left(\frac{x}{n}\right)\right] \cdot \frac{n}{x}$$

$$x y_1 = nb \cos\left[n\log\left(\frac{x}{n}\right)\right]$$

differentiating wrt x

$$x y_2 + (1) y_1 = nb\left[-\sin\left(n\log\left(\frac{x}{n}\right)\right)\right] \cdot n \cdot \frac{n}{x} \cdot \frac{1}{n}$$

$$x^2 y_2 + x y_1 = -n^2 b \sin\left(n \log\left(\frac{x}{n}\right)\right)$$

$$x^2 y_2 + x y_1 + n^2 y = 0 \quad \text{——— (1)}$$

by Leibnitz rule

$$\left[x^2 y_{n+2} + n(2x) y_{n+1} + \frac{n(n-1)}{2}(2) y_n\right] + \left[x y_{n+1} + n(1) y_n\right] + n^2 y_n = 0$$

$$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + 2n^2 y_n = 0$$

Ex-11 :- If $y = \sin[\log(x^2+2x+1)]$ prove that

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

Soln :-

$$y = \sin[\log(x^2+2x+1)]$$
$$= \sin[\log(x+1)^2]$$
$$= \sin[2\log(x+1)]$$

$$y_1 = \cos[2\log(x+1)] \cdot \frac{2}{x+1}$$

$$(x+1)y_1 = 2\cos[2\log(x+1)]$$

$$(x+1)y_2 + (1)y_1 = -2\sin[2\log(x+1)] \cdot \frac{2}{x+1}$$

$$(x+1)^2 y_2 + (x+1)y_1 = -4\sin[2\log(x+1)]$$

$$(x+1)^2 y_2 + (x+1)y_1 + 4y = 0$$

Applying leibnitz rule

$$\left[(x+1)^2 y_{n+2} + n\cdot 2(x+1)y_{n+1} + \frac{n(n-1)}{2}\cdot 2\cdot y_n\right] + \left[(x+1)y_{n+1} + n(1)y_n\right] + 4y_n = 0$$

$$(x+1)^2 y_{n+2} + 2n(x+1)y_{n+1} + n(n-1)y_n + (x+1)y_{n+1} + ny_n + 4y_n = 0$$

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

Hence proved

Ex :- $y = \sec^{-1}x$, prove that

$$x(x^2-1)y_{n+2} + [(2+3n)x^2-(n+1)]y_{n+1} + n(3n+1)xy_n + n^2(n-1)y_{n-1} = 0$$

Soln :- $y = \sec^{-1}x$

Differentiating wrt x

$$y_1 = \frac{1}{x\sqrt{x^2-1}}$$

$$x\sqrt{x^2-1}\, y_1 = 1$$

diff. again wrt x

$$x\sqrt{x^2-1}\, y_2 + \sqrt{x^2-1}\, y_1 + xy_1 \cdot \frac{1}{2\sqrt{x^2-1}} \cdot (2x) = 0$$

$$x(x^2-1)y_2 + (x^2-1)y_1 + x^2y_1 = 0$$

$$x(x^2-1)y_2 + (2x^2-1)y_1 = 0$$

By Leibnitz rule

$$\left[x(x^2-1)y_{n+2} + n(3x^2-1)y_{n+1} + \frac{n(n-1)}{2}(6x)y_n + \frac{n(n-1)(n-2)}{3!}(6)y_{n-1}\right]$$

$$+\left[(2x^2-1)y_{n+1} + n(4x)y_n + \frac{n(n-1)}{2!}(4)y_{n-1}\right] = 0$$

$$x(x^2-1)y_{n+2} + n(3x^2-1)y_{n+1} + 3xn(n-1)y_n + n(n-1)(n-2)y_{n-1}$$

$$+(2x^2-1)y_{n+1}+4nxy_n+2n(n-1)y_{n-1}=0$$

$$x(x^2-1)y_{n+2}+\left[3nx^2-n+2x^2-1\right]y_{n+1}+\left[3n^2x-3nx+4nx\right]y_n+n(n-1)\left[n-2+2\right]y_{n-1}=0$$

$$x(x^2-1)y_{n+2}+\left[(3n+2)x^2-(n+1)\right]y_{n+1}+n(3n+1)xy_n+n^2(n-1)y_{n-1}=0$$

Hence proved.

Ex:- If $x=\sin\theta$, $y=\sin 2\theta$, prove that

$$(1-x^2)y_{n+2}-(2n+1)x\,y_{n+1}-(n^2-4)y_n=0$$

Soln:- $y=\sin 2\theta = 2\sin\theta\cos\theta$

Now $\sin\theta = x$, $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$

$\therefore y = 2x\sqrt{1-x^2}$

Proceed as usual to get the answer.

Ex:- If $x=e^t$ and $y=\cos mt$ prove that

$$x^2y_{n+2}+(2n+1)x\,y_{n+1}+(m^2+n^2)y_n=0$$

Soln:- $y=\cos mt$

$x=e^t \Rightarrow t=\log x$

$\therefore y=\cos m(\log x) = \cos(m\log x)$

Ex:- If $x=\cos\theta$ and $\theta=\frac{1}{m}\log y$

Ex:- If $x = \cos\theta$ and $\theta = \frac{1}{m}\log y$

prove that $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2+m^2)y_n = 0$

Soln:- $x = \cos\theta = \cos\left(\frac{1}{m}\log y\right)$

$$\cos^{-1}x = \frac{1}{m}\log y$$

$$\log y = m\cos^{-1}x$$

$$y = e^{m\cos^{-1}x}$$