Problems using De-Moivre's Theorem

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n^{th} Derivative using De-Moivre's Theorem:

In some algebraic function n^{th} derivative can be put in an elegant form by using De-Moivre's Theorem. Also nth Derivatives of some inverse trigonometric functions can be put in a very compact form by using De-Moivre's Theorem.

De-Moivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Ex 1 :- If $y = \dfrac{1}{x^2+a^2}$, prove that

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \, \sin(n+1)\theta \quad \text{where } \theta = \tan^{-1}\left(\frac{a}{x}\right)$$

Soln :-

$$y = \frac{1}{x^2+a^2} = \frac{1}{(x+ai)(x-ai)} = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

By result $y = \dfrac{1}{ax+b}$ then $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$\therefore y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right] \quad \text{——— (1)}$$

Let $x = r\cos\theta$, $a = r\sin\theta$ so that

$r^2 = x^2+a^2$, $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{(r\cos\theta - i r\sin\theta)^{n+1}} = \frac{1}{r^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta]}$$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{[r\cos\theta - ir\sin\theta]^{n+1}} = \frac{1}{r^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta]}$$

$$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]$$

And $$\frac{1}{(x+ai)^{n+1}} = \frac{1}{[r\cos\theta + ir\sin\theta]^{n+1}} = \frac{1}{r^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta]}$$

$$= \frac{1}{r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

Substituting in (1)

$$y_n = \frac{(-1)^n n!}{2ai} \cdot \frac{1}{r^{n+1}} \cdot 2i\sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{a} \cdot \frac{1}{r^{n+1}} \sin(n+1)\theta$$

putting $r = \dfrac{a}{\sin\theta}$

$$y_n = \frac{(-1)^n n!}{a} \cdot \frac{\sin^{n+1}\theta}{a^{n+1}} \sin(n+1)\theta$$

$$y_n = \frac{(-1)^n n!}{a^{n+2}} \cdot \sin^{n+1}\theta \, \sin(n+1)\theta$$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

Ex2 :- If $y=\tan^{-1}x$, prove that

$$y_n = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta \quad \text{where } \theta=\tan^{-1}\left(\frac{1}{x}\right)$$

Sol^n :- $y=\tan^{-1}x$

Differentiating wrt x

$$y_1 = \frac{1}{1+x^2} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{1}{x-i}-\frac{1}{x+i}\right]$$

Differentiating $(n-1)$ times, by result

$$y=\frac{1}{ax+b},\quad y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

$$\therefore\ y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}\right] \quad \text{——— } (1)$$

put $x=r\cos\theta$, $1=r\sin\theta$ $\therefore r=\sqrt{x^2+1}$, $\theta=\tan^{-1}\left(\frac{1}{x}\right)$

$$\frac{1}{(x-i)^n} = \frac{1}{[r\cos\theta - ir\sin\theta]^n} = \frac{1}{r^n[\cos n\theta - i\sin n\theta]}$$

$$= \frac{1}{r^n}[\cos n\theta + i\sin n\theta]$$

$$\frac{1}{(x+i)^n} = \frac{1}{[r\cos\theta + i\sin\theta]^n} = \frac{1}{r^n}[\cos n\theta - i\sin n\theta]$$

Substituting in (1)

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n}\cdot 2i\sin n\theta\right]$$

$$= (-1)^{n-1}(n-1)!\cdot\frac{1}{r^n}\sin n\theta$$

put $r = \frac{1}{\sin\theta}$

$$y_n = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$$

where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Ex-3 :- If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ prove that

$$y_n = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta \quad \text{where } \theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Sol^n :- $y = \tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}(1) + \tan^{-1}(x) = \frac{\pi}{4} + \tan^{-1}(x)$

$$\left\{ \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1}(a) + \tan^{-1}(b) \right\}$$

Differentiating wrt x

$$y_1 = \frac{1}{x^2+1}$$

Proceeding as in ex(2), we will get the answer

Ex-4 : If $y = \sin^{-1}(2x$

Ex 4 If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ prove that

$y_n = 2(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$ where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

Soln:- $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

put $x = \tan\alpha \rightarrow \alpha = \tan^{-1}x$

$$y = \sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) = \sin^{-1}\left(\sin 2\alpha\right) = 2\alpha$$

$$\therefore y = 2\tan^{-1}x$$

Proceeding as in example (2), we will get the answer

Ex-5 $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$, prove that

$y_n = 2(-1)^{n-1}(n-1)! \sin^n\theta \sin n\theta$, $\underline{\theta = \tan^{-1}\left(\frac{1}{x}\right)}$

Soln:- $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right) = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$

Let $x = \tan\alpha$

$$y = \cos^{-1}\left(\frac{\tan^2\alpha - 1}{\tan^2\alpha + 1}\right) = \cos^{-1}\left(-\cos 2\alpha\right)$$

$$= \cos^{-1}\left[\cos(\pi + 2\alpha)\right]$$

$$y = \pi + 2\alpha$$

$$y = \pi + 2\tan^{-1}x$$

$$\therefore y_1 = \frac{2}{1+x^2}$$

proceeding as in ex.(2), we will get the answer.

Ex6 :- If $y = \dfrac{1}{x^2+x+1}$, prove that

$$y_n = \frac{2(-1)^n}{\sqrt{3}} \cdot \frac{n!}{r^{n+1}} \sin(n+1)\theta$$

where $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$ and $r = \sqrt{x^2+x+1}$

Solution $\quad y = \dfrac{1}{x^2+x+1} = \dfrac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$

let $x + \frac{1}{2} = X$

$$y = \frac{1}{X^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\left(X - i\frac{\sqrt{3}}{2}\right)\left(X + i\frac{\sqrt{3}}{2}\right)}$$

$$y = \frac{1}{\sqrt{3}i}\left[\frac{1}{\left(X - \frac{\sqrt{3}i}{2}\right)} - \frac{1}{\left(X + \frac{\sqrt{3}i}{2}\right)}\right]$$

By result : $y = \dfrac{1}{ax+b} \longrightarrow y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$\therefore y_n = \frac{1}{\sqrt{3}i}\left[\frac{(-1)^n n!}{\left(X - \frac{\sqrt{3}i}{2}\right)^{n+1}} - \frac{(-1)^n n!}{\left(X + \frac{\sqrt{3}i}{2}\right)^{n+1}}\right]$$

$$\frac{\ }{\sqrt{3}i}\left[\frac{\ }{(x-\frac{\sqrt{3}i}{2})^{n+1}} \quad \frac{\ }{(x+\frac{\sqrt{3}i}{2})^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{\sqrt{3}i}\left[\frac{1}{(x-\frac{\sqrt{3}i}{2})^{n+1}} - \frac{1}{(x+\frac{\sqrt{3}i}{2})^{n+1}}\right] \quad \text{--- } (1)$$

put $x = r\cos\theta$, $\frac{\sqrt{3}}{2} = r\sin\theta$

$$r = \sqrt{x^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{x^2+x+1} \quad \text{and} \quad \cot\theta = \frac{x}{\frac{\sqrt{3}}{2}} = \frac{2x}{\sqrt{3}}$$

$$\therefore \theta = \cot^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$\therefore \frac{1}{(x-\frac{\sqrt{3}i}{2})^{n+1}} = \frac{1}{r^{n+1}\left[\cos\theta - i\sin\theta\right]^{n+1}} = \frac{1}{r^{n+1}}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right]$$

$$\frac{1}{(x+\frac{\sqrt{3}i}{2})^{n+1}} = \frac{1}{r^{n+1}\left[\cos\theta + i\sin\theta\right]^{n+1}} = \frac{1}{r^{n+1}}\left[\cos(n+1)\theta - i\sin(n+1)\theta\right]$$

Substituting in (1)

$$y_n = \frac{(-1)^n n!}{\sqrt{3}i}\left[\frac{2i}{r^{n+1}}\sin(n+1)\theta\right]$$

$$y_n = \frac{2(-1)^n n!}{\sqrt{3}} \cdot \frac{1}{r^{n+1}}\sin(n+1)\theta$$

where $\theta = \cot^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$