Problems based on Trigonometric functions

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nth Derivative of Trigonometric Functions

In examples of this type we express product of trigonometric function as the sum and also the powers of sine and cosine by increasing the angle.

Formulae To be used

① $y = \sin(ax+b)$ then $y_n = a^n \sin\left[ax+b+n\frac{\pi}{2}\right]$

② $y = \cos(ax+b)$ then $y_n = a^n \cos\left[ax+b+n\frac{\pi}{2}\right]$

③ $y = e^{ax}\sin(bx+c)$ then

$$y_n = r^n e^{ax}\sin\left[bx+c+n\phi\right]$$

where $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

④ $y = e^{ax}\cos(bx+c)$ then

$$y_n = r^n e^{ax}\cos\left[bx+c+n\phi\right]$$

where $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

⑤ $y = k^x \sin(bx+c)$ then

$$y_n = r^n k^x \sin\left[bx+c+n\phi\right]$$

where $r = \sqrt{(\log k)^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{\log k}\right)$

⑥ $y = k^x \cos(bx+c)$ then

$$y_n = r^n k^x \cos\left[bx+c+n\phi\right]$$

where $r = \sqrt{(\log k)^2+b^2}$ $\phi = \tan^{-1}\left(\frac{b}{\log k}\right)$

Ex 1 :- Find n^{th} derivative of $\cos x \cos 2x \cos 3x$

Soln :- $y = \cos x \cos 2x \cos 3x$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$y = \cos x \cdot \frac{1}{2}[\cos 5x + \cos x]$$

$$= \frac{1}{2}[\cos x \cos 5x + \cos^2 x]$$

$$= \frac{1}{2}\left\{\frac{1}{2}(\cos 6x + \cos 4x) + \left(\frac{1+\cos 2x}{2}\right)\right\}$$

$$y = \frac{1}{4}[\cos 6x + \cos 4x + \cos 2x + 1]$$

By the result : $y = \cos(ax+b)$

$$y_n = a^n \cos\left(ax + b + n\frac{\pi}{2}\right)$$

Here $b = 0$

$$y_n = \frac{1}{4}\left[6^n \cos\left(6x + n\frac{\pi}{2}\right) + 4^n \cos\left(4x + n\frac{\pi}{2}\right) + 2^n \cos\left(2x + n\frac{\pi}{2}\right)\right]$$

Ex 2 :- Find n^{th} Derivative of $\cos^4 x$.

Soln :- $y = \cos^4 x = (\cos^2 x)^2 = \left[\frac{1+\cos 2x}{2}\right]^2$

$$\left[\cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$= \frac{1}{4}[1 + 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[\ldots \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{4}\left[1+2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

$$y = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$$

By the result: $y = \cos ax \rightarrow y_n = a^n \cos\left(ax + n\frac{\pi}{2}\right)$

$$\therefore y_n = \frac{1}{8}4^n \cos\left[4x + n\frac{\pi}{2}\right] + \frac{1}{2}2^n \cos\left[2x + n\frac{\pi}{2}\right]$$

Ex3 :- Find the nth derivative of $\sin^2 x \cos^3 x$

Soln :- $y = \sin^2 x \cos^3 x$

$$= (\sin x \cos x)^2 \cdot \cos x = \left(\frac{\sin 2x}{2}\right)^2 \cos x$$

$$= \frac{1}{4}\sin^2 2x \cos x$$

$\left[\sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$

$$= \frac{1}{8}\left[1 - \cos 4x\right]\cos x$$

$$= \frac{1}{8}\left[\cos x - \cos 4x \cos x\right]$$

$$= \frac{1}{8}\left[\cos x - \frac{1}{2}(\cos 5x + \cos 3x)\right]$$

$$= \frac{1}{8}\cos x - \frac{1}{16}\left[\cos 5x + \cos 3x\right]$$

$$y_n = \frac{1}{8}\cos\left(x + n\frac{\pi}{2}\right) - \frac{1}{16}\left[5^n \cos\left(5x + n\frac{\pi}{2}\right) + 3^n \cos\left(3x + n\frac{\pi}{2}\right)\right]$$

Ex4 :- $y = e^{5x}\cos x \cos 3x$ Find y_n

Soln :- $y = e^{5x}\cos x \cos 3x$

$$= e^{5x} \cdot \frac{1}{2}\left[\cos 4x + \cos 2x\right]$$

$$y = \frac{1}{2}\left[e^{5x}\cos 4x + e^{5x}\cos 2x\right]$$

By the result : $y = e^{ax}\cos bx$

$$y_n = r^n e^{ax}\cos(bx + n\phi)$$

$$r = \sqrt{a^2+b^2} \quad , \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$y_n = \frac{1}{2}\left[e^{5x} r_1^n \cos(4x + n\phi_1) + e^{5x} r_2^n \cos(2x + n\phi_2)\right]$$

$$r_1 = \sqrt{5^2+4^2} = \sqrt{41} \qquad \phi_1 = \tan^{-1}\left(\frac{4}{5}\right)$$

$$r_2 = \sqrt{5^2+2^2} = \sqrt{29} \qquad \phi_2 = \tan^{-1}\left(\frac{2}{5}\right)$$

$$y_n = \frac{e^{5x}}{2}\left[\left(\sqrt{41}\right)^n \cos\left[4x + n\tan^{-1}\left(\frac{4}{5}\right)\right] + \left(\sqrt{29}\right)^n \cos\left[2x + n\tan^{-1}\left(\frac{2}{5}\right)\right]\right]$$

Ex5 :- If $y = e^{x}(\sin x + \cos x)$

prove that $y_n = (\sqrt{2})^{n+1} e^{x}\sin\left[x + (n+1)\frac{\pi}{4}\right]$

Soln:- $y = e^x \sin x + e^x \cos x$

Using the result: $y = e^{ax} \sin bx$

$$y_n = r^n e^{ax} \sin(bx + n\phi)$$

$$r = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$y = e^{ax} \cos bx$$

$$y_n = r^n e^{ax} \cos(bx + n\phi)$$

$$r = \sqrt{a^2+b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Here $a = 1, \; b = 1$

$$\therefore y_n = r^n e^x \sin(x + n\phi) + r^n e^x \cos(x + n\phi)$$

$$r = \sqrt{1^2+1^2} = \sqrt{2} \qquad \phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$y_n = e^x \left[(\sqrt{2})^n \sin\left(x + \frac{n\pi}{4}\right) + (\sqrt{2})^n \cos\left(x + \frac{n\pi}{4}\right) \right]$$

$$= (\sqrt{2})^n e^x \left[\sin\left(x + \frac{n\pi}{4}\right) + \sin\left(\frac{\pi}{2} + x + n\frac{\pi}{4}\right) \right]$$

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

$$y_n = (\sqrt{2})^n e^x \; 2 \sin\left(x + n\frac{\pi}{4} + \frac{\pi}{4}\right) \cos\left(-\frac{\pi}{4}\right)$$

$$= (\sqrt{2})^n e^x \cdot 2 \sin\left(x + n\frac{\pi}{4} + \frac{\pi}{4}\right) \cdot \frac{1}{\sqrt{2}}$$

$$y = (\sqrt{2})^{n+1} e^x \sin\left(x + (n+1)\frac{\pi}{4}\right)$$

$$y_n = (\sqrt{2})^{n+1} e^x \sin\left[x + (n+1)\frac{\pi}{4}\right]$$

Ex If $y = \sin^7 x$ Find y_n

Soln :- Let $a = \cos x + i \sin x$ $\therefore \frac{1}{a} = \cos x - i \sin x$

$a^m = \cos mx + i \sin mx$ $\therefore \frac{1}{a^m} = \cos mx - i \sin mx$

$2i \sin x = a - \frac{1}{a}$ Also $a^m - \frac{1}{a^m} = 2i \sin mx$

$$(2i \sin x)^7 = \left(a - \frac{1}{a}\right)^7$$

$$= {}^7C_0 a^7 - {}^7C_1 a^6 \cdot \frac{1}{a} + {}^7C_2 a^5 \cdot \frac{1}{a^2} - {}^7C_3 a^4 \cdot \frac{1}{a^3}$$
$$+ {}^7C_4 a^3 \cdot \frac{1}{a^4} - {}^7C_5 a^2 \cdot \frac{1}{a^5} + {}^7C_6 a \cdot \frac{1}{a^6} - {}^7C_7 \cdot \frac{1}{a^7}$$

$$= a^7 - 7a^5 + 21a^3 - 35a + 35 \cdot \frac{1}{a} - 21 \cdot \frac{1}{a^3} + 7 \cdot \frac{1}{a^5} - \frac{1}{a^7}$$

$$2^7 i^7 \sin^7 x = \left(a^7 - \frac{1}{a^7}\right) - 7\left(a^5 - \frac{1}{a^5}\right) + 21\left(a^3 - \frac{1}{a^3}\right) - 35\left(a - \frac{1}{a}\right)$$

$$-2^7 i \sin^7 x = 2i \sin 7x - 7(2i \sin 5x) + 21(2i \sin 3x)$$
$$- 35(2i \sin x)$$

$$-2^6 \sin^7 x = \sin 7x - 7 \sin 5x + 21 \sin 3x - 35 \sin x$$

$$y = \sin^7 x = \frac{-1}{64}\left[\sin 7x - 7 \sin 5x + 21 \sin 3x - 35 \sin x\right]$$

$$y = \sin^7 x = \frac{-1}{64}\left[\sin 7x - 7\sin 5x + 21\sin 3x - 35\sin x\right]$$

If $y = \sin ax \rightarrow y_n = a^n \sin\left[ax + n\frac{\pi}{2}\right]$

$$\therefore y_n = \frac{-1}{64}\left[7^n \sin\left(7x + n\frac{\pi}{2}\right) - 7.5^n \sin\left(5x + n\frac{\pi}{2}\right) + 21\cdot 3^n \sin\left(3x + n\frac{\pi}{2}\right) - 35\sin\left(x + n\frac{\pi}{2}\right)\right]$$

H.W. Question

Find y_n for $y = \sin^5 x \cos^3 x = \underline{\sin^2 x}\ \underline{(\sin x \cos x)^3}$

Soln: $a = \cos x + i\sin x \qquad \frac{1}{a} = \cos x - i\sin x$

$a^m = \cos mx + i\sin mx \qquad \frac{1}{a^m} = \cos mx - i\sin mx$

$2i\sin x = a - \frac{1}{a}$ and $2\cos x = a + \frac{1}{a}$

$$(2i\sin x)^5 (2\cos x)^3 = \left(a - \frac{1}{a}\right)^5 \left(a + \frac{1}{a}\right)^3$$

$$= \left(a - \frac{1}{a}\right)^2 \left[\left(a + \frac{1}{a}\right)\left(a - \frac{1}{a}\right)\right]^3$$