SUCCESSIVE DIFFERENTIATION

INTRODUCTION:

If $f(x)$ is differentiable then higher order derivatives denote by
$f'(x), f''(x), f'''(x), f^{iv}(x), f^{v}(x), \ldots\ldots\ldots f^{n}(x), \ldots\ldots\ldots$
$y_1, y_2, y_3, \ldots\ldots, y_n, \ldots\ldots$ **or** $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots \frac{d^ny}{dx^n} \ldots\ldots$

$y = f(x)$
$y_1 = f'(x)$
$y_2 = f''(x)$
$y_n = f^n(x)$

The values of these derivatives at $x = a$ are denoted by $f^n(a), y_n(a)$ or $\left[\frac{d^ny}{dx^n}\right]_{x=a}$

DERIVATIVE OF N^TH ORDER:

1. If $y = (ax + b)^m$ then, $y_n = m(m-1)(m-2)\ldots(m-n+1)a^n(ax+b)^{m-n}$ if n < m.

Soln :- we have $y = (ax+b)^m$

differentiating $y_1 = m \cdot a\,(ax+b)^{m-1}$

differentiating $y_2 = m(m-1)\,a^2\,(ax+b)^{m-2}$

Similarly $y_3 = m(m-1)(m-2)\,a^3\,(ax+b)^{m-3}$

$\vdots$

$y_k = m(m-1)\cdots(m-(k-1))\,a^k\,(ax+b)^{m-k}$

In general $y_n = m(m-1)\cdots(m-(n-1))\,a^n\,(ax+b)^{m-n}$ if $n<m$

$\therefore\ y_n = m(m-1)\cdots(m-n+1)\,a^n\,(ax+b)^{m-n}$ if $n<m$

If $n=m$, $y_m = m(m-1)\cdots\cdots(m-m+1)a^m(ax+b)^{m-m}$
$= m(m-1)\cdots\cdots 1\,a^m = m!\,a^m$

if $n>m$, $y_n = 0$

for eg. $y = (ax+b)^4 \rightarrow y_1 = 4(ax+b)^3 \cdot a$
$y_2 = 12(ax+b)^2a^2$ $y_3 = 24(ax+b)\cdot a^3$
$y_4 = 24a^4$ $y_5 = 0$

2. If $y = (ax + b)^{-m}$ then, $y_n = (-1)^n(m)(m+1)(m+2)\ldots(m+n-1)a^n(ax+b)^{-m-n}$

Proof: Changing the sign of m in the above result,

$y = \frac{1}{\quad}$

2. If $y = (ax+b)^{-m}$ then, $y_n = (-1)^n (m)(m \cancel{+} 1)(m+2) \ldots (m+n-1) a^n (ax+b)^{-m-n}$

Proof: Changing the sign of m in the above result,

$y = \frac{1}{(ax+b)^m}$

$$y_n = (-m)(-m-1)(-m-2) \ldots\ldots (-m-n+1) a^n (ax+b)^{-m-n}$$
$$= (-1)^n (m)(m+1)(m+2) \ldots\ldots (m+n-1) a^n (ax+b)^{-m-n}$$
$$= (-1)^n (m+n-1)(m+n-2) \ldots\ldots (m+2)(m+1)m \frac{a^n}{(ax+b)^{m+n}}$$
$$= \frac{(-1)^n (m+n-1)(m+n-2) \ldots\ldots m(m-1)(m-2) \ldots\ldots 3 \cdot 2 \cdot 1}{(m-1)(m-2) \ldots\ldots 3 \cdot 2 \cdot 1} \frac{a^n}{(ax+b)^{m+n}}$$
$$\therefore y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}$$

$y_n = \frac{(-1)^n (m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}$

Corollary: **(i)** If $y = \frac{1}{x^m}$, then $y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{1}{x^{m+n}}$

(ii) If $y = x^m$, then

$$y_n = m(m-1)(m-2) \ldots (m-n+1) . x^{m-n} \text{ if } n < m$$
$$= m! \quad \text{if } n = m$$
$$= 0 \quad \text{if } n > m$$

3. If $y = \frac{1}{(ax+b)}$, then $y_n = \frac{(-1)^n \cdot n! \, a^n}{(ax+b)^{n+1}}$

Soln :- we have proved that If $y = \frac{1}{(ax+b)^m}$

then $y_n = \frac{(-1)^n (m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}$

If we put $m = 1$

$y = \frac{1}{ax+b}$ and $y_n = \frac{(-1)^n n!}{0!} \frac{a^n}{(ax+b)^{n+1}} = \frac{(-1)^n n! \, a^n}{(ax+b)^{n+1}}$

4. If $y = \log(ax+b)$ then $y_n = \frac{(-1)^{n-1} (n-1)! \, a^n}{(ax+b)^n}$

Soln :- In (3) we proved $y = \frac{1}{ax+b}$, $y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

Now if $y = \log(ax+b)$

$y_1 = \frac{1}{ax+b}$

differentiate $(n-1)$ times

$y_n = \frac{(-1)^{n-1} a^{n-1} (n-1)!}{(ax+b)^n}$

5. If $y = a^{mx}$ then $y_n = m^n a^{mx} (\log a)^n$

Sol^n :-

$$y = a^{mx}$$
$$y_1 = m \cdot a^{mx} \cdot (\log a)$$
$$y_2 = m^2 \cdot a^{mx} (\log a)^2$$

By generalization

$$y_n = m^n a^{mx} (\log a)^n$$

6. If $y = e^{mx}$ then $y_n = m^n e^{mx}$

Sol^n :-

$$y = e^{mx}$$
$$y_1 = m e^{mx}$$
$$y_2 = m^2 e^{mx}$$
$$\vdots$$

generalizing

$$y_n = m^n e^{mx}$$

7. If $y = \sin(ax + b)$ then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

Sol^n :-

$$y = \sin(ax+b)$$
$$y_1 = a \cos(ax+b) = a \sin\left(ax+b+\frac{\pi}{2}\right)$$
$$y_2 = a^2 \cos\left(ax+b+\frac{\pi}{2}\right) = a^2 \sin\left[ax+b+\frac{2\pi}{2}\right]$$
$$y_3 = a^3 \cos\left(ax+b+\frac{2\pi}{2}\right) = a^3 \sin\left[ax+b+\frac{3\pi}{2}\right]$$
$$\vdots$$

Generalizing this

$$y_n = a^n \sin\left[ax+b+\frac{n\pi}{2}\right]$$

In particular If $y = \sin x$

$$y_n = \sin\left(x + \frac{n\pi}{2}\right)$$

8. If $y = cos\ (ax + b)$ then $y_n = a^n cos\ \left(ax + b + \frac{n\pi}{2}\right)$

similarly $y = \cos(ax+b)$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

9. If $y = e^{ax} sin\ (bx + c)$ then $y_n = r^n e^{ax} sin\ (bx + c + n\Phi)$

Where $r = \sqrt{a^2 + b^2}$ and $\Phi = tan^{-1}\left(\frac{b}{a}\right)$

<u>Soln</u> :- $y = e^{ax} \sin(bx+c)$

$$y_1 = e^{ax} \cdot a \sin(bx+c) + e^{ax} \cdot b \cos(bx+c)$$

$$= e^{ax}\left[a \sin(bx+c) + b\cos(bx+c)\right]$$

$$= e^{ax} \cdot \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}} \sin(bx+c) + \frac{b}{\sqrt{a^2+b^2}} \cos(bx+c)\right]$$

If we write $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$

also we put $r = \sqrt{a^2+b^2}$, $\tan\phi = \frac{b}{a}$

$$y_1 = re^{ax}\left[\cos\phi \sin(bx+c) + \sin\phi \cos(bx+c)\right]$$

$$y_1 = re^{ax}\left[\sin(bx+c+\phi)\right]$$

we observe that y_1 is what y becomes when y is multiplied by r and angle is increased by ϕ

By same reasoning $y_2 = r^2 e^{ax} \sin(bx+c+2\phi)$

$$y_3 = r^3 e^{ax} \sin(bx+c+3\phi)$$

By generalization $y_n = r^n e^{ax} \sin(bx + c + n\phi)$

corollary :- If $c = 0$ ie $y = e^{ax} \sin bx$

then $y_n = r^n e^{ax} \sin(bx + n\phi)$

where $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

10. If $y = e^{ax}\cos(bx + c)$ then $y_n = r^n e^{ax}\cos(bx + c + n\Phi)$ $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Corollary: If $c = 0$, i.e. if $y = e^{ax}\cos bx$ then $y_n = r^n e^{ax}\cos(bx + n\Phi)$

11. If $y = k^x \sin(bx + c)$ then $y_n = r^n k^x \sin(bx + c + n\Phi)$

Where $r = \sqrt{(\log k)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\frac{b}{\log k}\right)$

Soln :- $y = k^x \sin(bx+c)$

$= e^{x \log k} \sin(bx+c) = e^{ax} \sin(bx+c)$

where $a = \log k$

using result (9)

$y_n = r^n e^{ax} \sin(bx+c+n\phi) = r^n k^x \sin(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2} = \sqrt{(\log k)^2+b^2}$

$\phi = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{\log k}\right)$

12. If $y = k^x \cos(bx + c)$ then $y_n = r^n k^x \cos(bx + c + n\Phi)$

Where $r = \sqrt{(\log k)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\frac{b}{\log k}\right)$

	FUNCTION	N^TH ORDER DERIVATIVE
1	$y = (ax + b)^m$ where $m \in N$	If $n < m$ $y_n = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$ If $n = m$, $y_n = m!\, a^n$ If $n > m$, $y_n = 0$
2	$y = (ax + b)^{-m}$	$y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{a^n}{(ax+b)^{m+n}}$
3	$y = \frac{1}{(ax+b)}$	$y_n = \frac{(-1)^n \cdot n!\, a^n}{(ax+b)^{n+1}}$

4	$y = \log{(\text{ax} + \text{b})}$	$y_n = \dfrac{(-1)^{n-1}\,(n-1)!\ a^n}{(ax+b)^n}$
5	$y = a^{mx}$	$y_n = m^n a^{mx} (log\ a)^n$
6	$y = e^{mx}$	$y_n = m^n e^{mx}$
7	$y = sin\ (ax+b)$	$y_n = a^n sin\ \left(ax + b + \dfrac{n\pi}{2}\right)$
8	$y = cos\ (ax+b)$	$y_n = a^n cos\ \left(ax + b + \dfrac{n\pi}{2}\right)$
9	$y = e^{ax} sin\ (bx+c)$	$y_n = r^n e^{ax} sin\ (bx + c + n\Phi)$ $r = \sqrt{a^2+b^2}\ \ \&\ \ \Phi = tan^{-1}\left(\frac{b}{a}\right)$
10	$y = e^{ax} cos\ (bx+c)$	$y_n = r^n e^{ax} cos\ (bx + c + n\Phi)$ $r = \sqrt{a^2+b^2}\ \ \&\ \ \Phi = tan^{-1}\left(\frac{b}{a}\right)$
11	$y = k^x \sin(bx+c)$	$y_n = r^n k^x sin\ (bx + c + n\Phi)$ $r = \sqrt{\left(\log k\right)^2 + b^2}\ \&\ \Phi = tan^{-1}\left(\frac{b}{\log k}\right)$
12	$y = k^x \cos(bx+c)$	$y_n = r^n k^x cos\ (bx + c + n\Phi)$ $r = \sqrt{\left(\log k\right)^2 + b^2}\ \&\ \ \Phi = tan^{-1}\left(\frac{b}{\log k}\right)$