- **Case (iii) When the r.h.s. $X = x^m$ where m is a positive integer**
- When $X = x^m$, we write f(D) in ascending powers of D by putting it in the form $1 + F(D)$.
- Then, $P.I = \frac{1}{f(D)} x^m = \frac{1}{1+\emptyset(D)} x^m = [1 + \emptyset(D)]^{-1} x^m$
- By expanding the bracket by the formula,
- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots$ **or**
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$ **or**
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots\ldots$ **or**
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$

$$\frac{1}{D^2+D+1} x^3 = \frac{1}{1+(D^2+D)} x^3 \qquad [1+\phi(D)]^{-1}$$

$$= [1+(D^2+D)]^{-1} x^3$$

$$(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \ldots$$

$$= [1 - (D^2+D) + (D^2+D)^2 - (D^2+D)^3 + (D^2+D)^4 \ldots] x^3$$

$$= [1 - D^2 - D + \cancel{D^4} + 2D^3 + D^2 - \cancel{D^6} - 3\cancel{D^5} - 3\cancel{D^4} - D^3] x^3$$

$$= x^3 - 6x - 3x^2 + 12 + 6x - 6 = x^3 - 3x^2 + 6$$

- and by operating each term of the expansion on xm we get the required Particular Integral.
- It is obvious that in the expansion, terms beyond the m^th^ power of D need not be written
- since the derivatives of x^m of powers higher than m are zero.

- EXAMPLE-1: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 3x^2 - 5x + 2$ $\rightarrow (D^3 - 2D + 4)y = 3x^2 - 5x + 2$

<u>Solution</u> A.E is $m^3 - 2m + 4 = 0$

$$m = -2,\ 1 \pm i$$

$\therefore$ The C.F is $y_c = c_1 e^{-2x} + e^{x}[c_2 \cos x + c_3 \sin x]$

The PI is $y_p = \dfrac{1}{D^3 - 2D + 4}(3x^2 - 5x + 2)$

$$= \frac{1}{4} \cdot \frac{1}{1 + \left(\frac{D^3 - 2D}{4}\right)}(3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \left(\frac{D^3 - 2D}{4}\right)\right]^{-1}(3x^2 - 5x + 2)$$

(wkt $(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \cdots\cdots$)

$$y_p = \frac{1}{4} \cdot \left[1 - \left(\frac{D^3 - 2D}{4}\right) + \left(\frac{D^3 - 2D}{4}\right)^2 - \left(\frac{D^3 - 2D}{4}\right)^3 + \cdots\cdots\right](3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[1 + \frac{2D}{4} + \frac{4D^2}{16}\right](3x^2 - 5x + 2)$$

$$= \frac{1}{4}\left[(3x^2 - 5x + 2) + \frac{1}{2}(6x - 5) + \frac{1}{4}(6)\right]$$

$$y_p = \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$$

The complete Solution is

$$y = y_c + y_p = c_1 e^{-2x} + e^{x}[c_2 \cos x + c_3 \sin x] + \frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{4}$$

EXAMPLE-2: $(D^3 - D^2 - 6D)y = x^2 + 1$

Soln: A.E is $m^3 - m^2 - 6m = 0$

$$m(m^2 - m - 6) = 0$$

$$m = 0, -2, 3$$

C.F is $y_c = c_1 + c_2 e^{-2x} + c_3 e^{3x}$

PI is $y_p = \dfrac{1}{D^3 - D^2 - 6D}(x^2+1)$

$$= -\frac{1}{6D} \cdot \frac{1}{1+\left(\frac{D-D^2}{6}\right)}(x^2+1)$$

$$= \frac{-1}{6D} \cdot \left[1+\left(\frac{D-D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$y_p = \frac{-1}{6D}\left[1-\left(\frac{D-D^2}{6}\right)+\left(\frac{D-D^2}{6}\right)^2 \cdots\cdots\right](x^2+1)$$

$$= \frac{-1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\right](x^2+1)$$

$$= \frac{-1}{6D}\left[x^2+1-\frac{1}{6}(2x)+\frac{2}{6}+\frac{2}{36}\right]$$

$$= \frac{-1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\int\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)dx$$

$$y_p = -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

$\therefore$ The complete solution is

$$y = y_c + y_p = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25x}{18}\right]$$

EXAMPLE-3: $\dfrac{d^3y}{dt^3}+\dfrac{dy}{dt} = \cos t + t^2 + 3 \rightarrow (D^3+D)y = \cos t + t^2 + 3$

Sol^n:- A.E. $m^3 + m = 0$

$m = 0, \pm i$

The C.F is $y_c = c_1 + c_2 \cos t + c_3 \sin t$

$$y_p = \frac{1}{D^3+D}\left(\cos t + t^2 + 3\right)$$

$$= \frac{1}{D^3+D}(\cos t) + \frac{1}{D^3+D}(t^2+3)$$

$$\frac{1}{D^3+D}\cos t = \frac{t}{3D^2+1}\cos t = \frac{-t}{2}\cos t$$

$D^2 = -1^2$ deno $= 0$ put $D^2 = -1$

$$\frac{1}{D^3+D}(t^2+3) = \frac{1}{D[1+D^2]}(t^2+3)$$

$$= \frac{1}{D}\cdot\left[1+D^2\right]^{-1}(t^2+3)$$

$$= \frac{1}{D}\left[1 - D^2 + D^4 - D^6 \cdots\right](t^2+3)$$

$$= \frac{1}{D}(1-D^2)(t^2+3)$$

$$= \frac{1}{D}(t^2+3-2)$$

$$= \frac{1}{D}(t^2+1)$$

$$= \int (t^2+1)\,dt$$

$$= \frac{t^3}{3} + t$$

$$\therefore y_p = -\frac{t}{2}\cos t + \frac{t^3}{3} + t$$

$$\therefore y = y_c + y_p = c_1 + c_2 \cos t + c_3 \sin t - \frac{t}{2}\cos t + \frac{t^3}{3} + t$$

<u>Another way</u>

Another way

$$\frac{1}{D^3+D}\cos t = \frac{1}{D[1+D^2]}\cos t = \frac{1}{D}\cdot\frac{1}{1+D^2}\cos t$$

put $D^2=-1$

$D^2+1=0$

$$= \frac{1}{D}\cdot\frac{t}{2D}\cos t$$

$$= \frac{1}{D}\cdot\frac{t}{2}\sin t$$

$$= \frac{1}{2}\int t\sin t$$

$$\frac{1}{D^3+D}\cos t \qquad = \frac{1}{2}\left[-t\cos t + \sin t\right]$$

$$\therefore \quad y = c_1 + c_2\cos t + c_3\sin t - \frac{t}{2}\cos t + \frac{1}{2}\sin t + \frac{t^3}{3} + t$$