$D(e^{ax}) = a e^{ax}$

$D(\sin ax) = a \cos ax$

$D^2(\sin ax) = -a^2 \sin ax$

METHOD TO FIND PI WHEN RHS= $\sin \boldsymbol{ax}$, $\cos \boldsymbol{ax}$

- **Case (ii) When the r.h.s.** $X = \boldsymbol{\sin ax,\ \cos ax}$
- $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax,\ \ if\ f(-a^2) \neq 0$ And

$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax,\ \ if\ f(-a^2) \neq 0$

put $D^2 = -a^2$

$$\frac{1}{D^4+3D^2+2}\sin 2x = \frac{1}{(-4)^2+3(-4)+2}\sin 2x = \frac{1}{16-12+2}\sin 2x = \frac{\sin 2x}{6}$$

put $D^2 = -2^2 = -4$

- When $f(-a^2) = 0\ then\ \frac{1}{f(D^2)}\sin ax = \frac{x}{f'(-a^2)}\sin ax,\ \ If\ f'(-a^2) \neq 0$ And
- When $f(-a^2) = 0\ then\ \frac{1}{f(D^2)}\cos ax = \frac{x}{f'(-a^2)}\cos ax,\ \ If\ f'(-a^2) \neq 0$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2D}\cos ax = \frac{x}{2}\int \cos ax\, dx = \frac{x \sin ax}{2a}$$

put $D^2 = -a^2$

$EXAMPLE - 1$: $(D^3 + D^2 + D + 1)y = \sin^2 x$

Soln :- A.E. $m^3+m^2+m+1=0 \rightarrow m^2(m+1)+1(m+1)=0$

$(m+1)(m^2+1)=0$

$m=-1,\ m^2=-1$

$m=-1,\ m=\pm i$

$m = -1, \pm i$

$\therefore$ C.F is

$$y_c = c_1 e^{-x} + e^{0x}[c_2 \cos x + c_3 \sin x]$$

$$y_c = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$$

Now $PI = y_p = \frac{1}{D^3+D^2+D+1}\sin^2 x$

$$= \frac{1}{D^3+D^2+D+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$\frac{}{D^3+D^2+D+1}\left(\frac{}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D^3+D^2+D+1}e^{0x} - \frac{1}{2}\frac{1}{D^3+D^2+D+1}\cos 2x$$

(put $D=0$) (put $D^2=-4$)

$$= \frac{1}{2}\cdot\frac{1}{0+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-4D+(-4)+D+1}\cos 2x$$

$$y_p = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{-3(D+1)}\cos 2x$$

$$= \frac{1}{2} + \frac{1}{6}\cdot\frac{D-1}{D^2-1}\cos 2x$$

(put $D^2=-4$)

$$= \frac{1}{2} + \frac{1}{6}\cdot\frac{(D-1)}{-5}\cos 2x$$

$$= \frac{1}{2} - \frac{1}{30}(D-1)\cos 2x$$

$$= \frac{1}{2} - \frac{1}{30}\left[D(\cos 2x) - \cos 2x\right]$$

$$= \frac{1}{2} - \frac{1}{30}\left[-2\sin 2x - \cos 2x\right]$$

$$y_p = \frac{1}{2} + \frac{2\sin 2x + \cos 2x}{30}$$

$\therefore$ complete solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{2} + \frac{2\sin 2x+\cos 2x}{30}}$$

EXAMPLE-2: $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 27y = \cos 3x$

Solⁿ:- $(D^3 - 3D^2 + 9D - 27)y = \cos 3x$

A.E is $m^3 - 3m^2 + 9m - 27 = 0 \rightarrow m^2(m-3) + 9(m-3) = 0$

$(m-3)(m^2+9) = 0$

$m = 3, \pm 3i$

$m = 3, 3i, -3i$

$\therefore$ C.F is

$$y_c = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x$$

$$PI = y_p = \frac{1}{D^3 - 3D^2 + 9D - 27} \cos 3x$$

put $D^2 = -3^2 = -9$, Denominator $= 0$

$$= \frac{x}{3D^2 - 6D + 9} \cos 3x$$

put $D^2 = -9$

$$= \frac{x}{-6D - 18} \cos 3x = -\frac{x}{6} \cdot \frac{1}{D+3} \cos 3x$$

$$= -\frac{x}{6} \cdot \frac{D-3}{D^2 - 9} \cos 3x$$

put $D^2 = -9$

$$= -\frac{x}{6} \cdot \frac{D-3}{-18} \cos 3x$$

$$= \frac{x}{108} \left[D(\cos 3x) - 3\cos 3x \right]$$

$$= \frac{x}{108} \left[-3\sin 3x - 3\cos 3x \right]$$

$$\therefore y_p = \frac{-x}{36} (\sin 3x + \cos 3x)$$

$$\therefore \boxed{y = y_c + y_p = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36} [\sin 3x}$$

$$\therefore \boxed{y = y_c + y_p = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{36}\left[\sin 3x + \cos 3x\right]}$$

EXAMPLE-3: $(D^4 - 1)y = e^x + \cos x \cos 3x$

Soln: A.E is $m^4 - 1 = 0$

$$(m^2+1)(m^2-1) = 0$$

$$m = 1, -1, i, -i$$

$\therefore$ CF is $y_c = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

PI is $y_p = \frac{1}{D^4-1}\left[e^x + \frac{1}{2}\left(\cos 2x + \cos 4x\right)\right]$

$$= \frac{1}{D^4-1} e^x + \frac{1}{2}\frac{1}{D^4-1}\cos 2x + \frac{1}{2}\frac{1}{D^4-1}\cos 4x$$

put $D=1$, $f(D)=0$ put $D^2=-4$ put $D^2=-16$

$$= \frac{x}{4D^3} e^x + \frac{1}{2}\cdot\frac{1}{16-1}\cos 2x + \frac{1}{2}\cdot\frac{1}{256-1}\cos 4x$$

$$y_p = \frac{xe^x}{4} + \frac{\cos 2x}{30} + \frac{\cos 4x}{510}$$

$$\therefore y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{xe^x}{4} + \frac{\cos 2x}{30} + \frac{\cos 4x}{510}$$

Example-4: $\frac{d^2y}{dx^2} + y = \sin x \sin 2x + 2^x$

Soln: $(D^2+1)y = \sin x \sin 2x + 2^x$

AE is $m^2+1=0$ $m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

PI. $y_p = \frac{1}{D^2+1}\left[\sin x \sin 2x + 2^x\right]$

$= \frac{1}{D^2+1}\left[\frac{1}{2}(\cos x - \cos 3x) + e^{x\log 2}\right]$

$= \frac{1}{2} \cdot \frac{1}{D^2+1}\cos x - \frac{1}{2}\frac{1}{D^2+1}\cos 3x + \frac{1}{D^2+1}e^{x\log 2}$

put $D^2=-1$ put $D^2=-9$ put $D=\log 2$

$D^2+1=0$

$= \frac{1}{2} \cdot \frac{x}{2D}\cos x - \frac{1}{2} \cdot \frac{1}{-8}\cos 3x + \frac{1}{(\log 2)^2+1}e^{x\log 2}$

$= \frac{x}{4}\int \cos x \, dx + \frac{1}{16}\cos 3x + \frac{2^x}{(\log 2)^2+1}$

$= \frac{x\sin x}{4} + \frac{\cos 3x}{16} + \frac{2^x}{(\log 2)^2+1}$

$\therefore$ $y = y_c + y_p$

$= c_1 \cos x + c_2 \sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16} + \frac{2^x}{(\log 2)^2+1}$