PI WHEN RHS= e^{ax}

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METHOD TO FIND PI

$f(D)y = X$ where X is some function of x

complete solution = complementary Function (C.F.) + particular Integral (PI)

$$y = y_c + y_p$$

To find y_c, we take $f(D)y = 0$

Auxillary eqn $f(m) = 0$

find roots of A.E. and write y_c.

$D \to \frac{d}{dx}$

$\frac{1}{D} \to \int dx$

To find y_p, $f(D)y = X \Rightarrow y_p = \frac{1}{f(D)} X$

This depends on X

- Case (i) **When the r.h.s. $X = e^{ax}$.**
- $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$

$$\frac{1}{D^3+D^2+D} e^{2x} = \frac{1}{2^3+2^2+2} e^{2x} = \frac{1}{8+4+2} e^{2x} = \frac{e^{2x}}{14}$$

- When $f(a) = 0$, $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ If $f'(a) \neq 0$

$\frac{1}{D^2-1} e^{x}$ if we put $D=1$, $D^2-1=0$

$$= x \cdot \frac{1}{2D} e^{x} = x \cdot \frac{1}{2} e^{x} = \frac{x e^{x}}{2}$$

- When $f'(a) = 0$, $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$ If $f''(a) \neq 0$

$\frac{1}{(D+2)^2} e^{-2x}$ if we put $D=-2$, $(D+2)^2 = 0$

$= x \cdot \frac{1}{\quad} e^{-2x}$ if we put $D=-2$, $D+2=0$

$$2(D+2)$$

$$= x^2 \cdot \frac{1}{2} e^{-2x} = \frac{x^2 e^{-2x}}{2}$$

- When $f''(a) = 0$, $\frac{1}{f(D)} e^{ax} = x^3 \frac{1}{f'''(a)} e^{ax}$ If $f'''(a) \neq 0$ etc

- EXAMPLE -1: $(D^3 - 2D^2 - 5D + 6)\, y = (e^{2x} + 3)^2$

Soln :- Associated eqn is $(D^3 - 2D^2 - 5D + 6)\, y = 0$

Auxillary eqn is $m^3 - 2m^2 - 5m + 6 = 0$

$$m = -2, 1, 3$$

$\therefore$ C.F is $y_c = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x}$

Now $y_p = \frac{1}{D^3 - 2D^2 - 5D + 6} (e^{2x} + 3)^2$

$$= \frac{1}{D^3 - 2D^2 - 5D + 6} (e^{4x} + 6e^{2x} + 9)$$

$$= \underset{(D=4)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{4x} + 6 \underset{(D=2)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{2x}$$

$$+ 9 \underset{(D=0)}{\frac{1}{D^3 - 2D^2 - 5D + 6}} e^{0x}$$

$$y_p = \frac{e^{4x}}{18} - \frac{3}{2}e^{2x} + \frac{3}{2}$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x} + c_3 e^{3x} + \frac{e^{4x}}{18} - \frac{3e^{2x}}{2} + \frac{3}{2}}$$

EXAMPLE-2: $6\frac{d^2y}{dx^2} + 17\frac{dy}{dx} + 12y = e^{-3x/2} + 2^x$

Sol^n :- $(6D^2 + 17D + 12)y = e^{-3x/2} + e^{x\log 2}$

A.E is $6m^2 + 17m + 12 = 0$

$$m = -\frac{4}{3}, -\frac{3}{2}$$

$\therefore$ C.F is $y_c = c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x}$

Now P.I. $= y_p = \frac{1}{6D^2 + 17D + 12}\left(e^{-\frac{3}{2}x} + e^{x\log 2}\right)$

$$= \frac{1}{6D^2 + 17D + 12} e^{-\frac{3}{2}x} + \frac{1}{6D^2 + 17D + 12} e^{x\log 2}$$

$(D = -\frac{3}{2})$ $f(D) = 0$ $\quad$ $(D = \log 2)$

$$= x \cdot \frac{1}{12D + 17} e^{-\frac{3}{2}x} + \frac{1}{6(\log 2)^2 + 17(\log 2) + 12} e^{x\log 2}$$

$(D = -\frac{3}{2})$

$$= -xe^{-\frac{3}{2}x} + \frac{2^x}{6(\log 2)^2 + 17(\log 2) + 12}$$

$\therefore$ complete soln is

$y = y_c + y_p$

$$= c_1 e^{-\frac{4}{3}x} + c_2 e^{-\frac{3}{2}x} - xe^{-\frac{3}{2}x} + \frac{2^x}{6(\log 2)^2 + 17(\log 2) + 12}$$

EXAMPLE-3: $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cos h^2 2x$

Soln: $(D^3 - 4D)y = 2\left[\frac{e^{2x} + e^{-2x}}{2}\right]^2 = \frac{1}{2}\left[e^{4x} + e^{-4x} + 2\right]$

A.E is $m^3 - 4m = 0$

$m(m^2 - 4) = 0 \qquad m = 0, 2, -2$

$\therefore$ C.F. is $y_c = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$

$$= c_1 + c_2 e^{2x} + c_3 e^{-2x}$$

$\therefore$ P.I. $= y_p = \frac{1}{D^3 - 4D} \cdot \frac{1}{2}\left(e^{4x} + e^{-4x} + 2\right)$

$$= \frac{1}{2} \cdot \frac{1}{D^3 - 4D} e^{4x} + \frac{1}{2} \frac{1}{D^3 - 4D} e^{-4x} + \frac{1}{D^3 - 4D} \cdot e^{0x}$$

$(D=4) \qquad (D=-4) \qquad (D=0)$

$f(D) = 0$

$$= \frac{1}{2} \cdot \frac{e^{4x}}{48} + \frac{1}{2} \cdot \frac{e^{-4x}}{(-48)} + x \cdot \frac{e^{0x}}{3D^2 - 4}$$

$(D=0)$

$$= e^{4x} \qquad e^{-4x} - xe^{0x}$$

$$= \frac{e^{4x}}{96} - \frac{e^{-4x}}{96} - \frac{x e^{0x}}{4}$$

$$= \frac{1}{48}\left[\frac{e^{4x} - e^{-4x}}{2}\right] - \frac{x}{4}$$

$$y_p = \frac{\sinh 4x}{48} - \frac{x}{4}$$

$\therefore$ The complete soln is

$$y = y_c + y_p = c_1 + c_2 e^{2x} + c_3 e^{-2x} + \frac{\sinh 4x}{48} - \frac{x}{4}$$

Ex 4 $(D^2 - D - 6)y = e^x \cosh 2x$

Soln: A.E is $m^2 - m - 6 = 0$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

$\therefore$ C.F is $y_c = c_1 e^{3x} + c_2 e^{-2x}$

Now P.I. $= y_p = \frac{1}{D^2 - D - 6}(e^x \cosh 2x)$

$$= \frac{1}{D^2 - D - 6}\left[e^x \cdot \frac{1}{2}\left(e^{2x} + e^{-2x}\right)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - D - 6} e^{3x} + \frac{1}{2} \frac{1}{D^2 - D - 6} e^{-x}$$

$D = 3$ $\qquad$ $D = -1$

$f(D) = 0$

$$= \frac{x}{2} \cdot \frac{1}{2D-1} e^{3x} + \frac{1}{2} \cdot \frac{1}{1+1-6} e^{-x}$$

(D=3)

$$= \frac{x}{2} \cdot \frac{1}{5} e^{3x} + \frac{1}{2} \cdot \frac{e^{-x}}{-4}$$

$$y_p = \frac{x e^{3x}}{10} - \frac{e^{-x}}{8}$$

$\therefore$ The complete soln is

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-2x} + \frac{x e^{3x}}{10} - \frac{e^{-x}}{8}$$