Higher Order Linear DE

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degree = 1. order > 1

HIGHER ORDER DIFFERENTIAL EQUATION

- **Definition:** An equation of the form $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \ \ldots . + P_n y = X$ ……(1) Where $P_1, P_2, \ldots \ldots \ldots\ P_n$ are constants and X is a function of x only is called a **linear differential equation with constant coefficients**.

For example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = \cos x$$

$$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 7y = \sinh^2 x$$

THE OPERATOR D

- Let D be the symbol which denotes differentiation with respect to x, say, of the function which immediately follows it i.e. D stands for $\frac{d}{dx}$.
- Thus, if y is a differentiable function of x then $D(y) = \frac{d}{dx}(y)\ or\ Dy = \frac{dy}{dx}$,
- $D(Dy) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ $\frac{d^2y}{dx^2} + 4y = e^x \rightarrow D^2y + 4y = e^x$
- Let us further denote the operation of D repeated twice, thrice, ……. n times by $D^2, D^3, \ldots \ldots \ldots\ D^n$.
- With this notation, $D^2y = \frac{d^2y}{dx^2}, D^3y = \frac{d^3y}{dx^3}, \ldots \ldots . D^n y = \frac{d^n y}{dx^n}$
- From this point of view the symbol D is called as operator and the function y on which it **operates** is called **operand**.
- With this notation the differential equation (1) can be written as
- $D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + . . + P_n y = X$
- i.e. $\left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ \ldots . + P_n\right) y = X$ i.e. $\boxed{f(D)y = X}$

$$\frac{d^5y}{dx^5} - 4\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - 5y = e^x$$

$$D^5y - 4D^4y + 3D^3y - D^2y + Dy - 5y = e^x$$

$$(D^5 - 4D^4 + 3D^3 - D^2 + D - 5)y = e^x$$

$$f(D)y = e^x$$

METHOD TO FIND COMPLETE SOLUTION

$y = y_c + y_p$

- Complete Solution = Complementary Function (C.F.) + Particular integral (P.I.).
- (1) Write the given differential equation in the form $f(D)y = X$
- **(2)** Write the associated equation $f(D)y = 0$ $\rightarrow$ y_c
- **(3)** Write the **auxiliary equation** by putting $D = m$ in the terms within bracket when the equation is written in the symbolic form as $f(m) = 0$
- $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = 0$
- Auxiliary equation is $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n = 0$
  It is an equation of n^th degree in m having n roots says $m_1, m_2, m_3 \ldots\ldots m_n$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^x$$

$$D^3y + 3D^2y + 3Dy + y = e^x$$

$$(D^3 + 3D^2 + 3D + 1)\,y = e^x \rightarrow f(D)y = X$$

Associated eqn $f(D)y = 0$ ie $(D^3 + 3D^2 + 3D + 1)\,y = 0$

Auxillary eqn is $f(m) = 0$ ie $m^3 + 3m^2 + 3m + 1 = 0$

- **(4)** Write the **complementary function (C.F)** as follows:
- **Case (i)** when roots are **real and different** $m_1, m_2, \ldots, m_n$
- The C.F. is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$

$$(D^2 - 3D + 2)\,y = 0$$

Aux $\rightarrow$ $m^2 - 3m + 2 = 0$ $(m-2)(m-1) = 0$ $\Rightarrow m = 1, 2$

$$\text{C.F.} \Rightarrow y_c = c_1 e^{x} + c_2 e^{2x}$$

- **Case (ii)** When roots are **real and equal** i.e. repeated
- **(a)** Suppose the auxiliary equation has got two equal roots. Say, each m_1 and Let the other roots be $m_3, m_4 \ldots\ldots m_n$ then the C.F is
- $y = (C_1 + C_2 x)e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$

for eg. $m = 1, 1, 2, 3, -6$

$$y_c = (c_1 + c_2 x)e^{x} + c_3 e^{2x} + c_4 e^{3x} + c_5 e^{-6x}$$

- **(b)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and let the other roots be $m_4, m_5 \ldots\ldots\ldots\ldots m_n$ then the C.F. is

- **(b)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and let the other roots be $m_4, m_5 \ldots\ldots\ldots.. m_n$ then the C.F. is
- $y = (C_1 + C_2x + C_3x^2)e^{m_1x} + C_4e^{m_4x} + \ldots. + C_ne^{m_nx}$

for eg. $m = 2, 2, 2, -1, -3, 7$

$y_c = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x} + c_5e^{-3x} + c_6e^{7x}$

- **(c)** Suppose the auxiliary equation has got three equal roots. Say, each m_1, and next two equal roots say, each m_2, and let the other roots be $m_6, m_7 \ldots\ldots\ldots\ldots.. m_n$ then the C.F is

$$y = (C_1 + C_2x + C_3x^2)e^{m_1x} + (C_4 + C_5x)e^{m_2x} + C_6e^{m_6x} + \ldots\ldots. + C_ne^{m_nx}$$

for eg. $m = 3, 3, 3, -1, -1, 2, 5$

$y_c = (c_1 + c_2x + c_3x^2)e^{3x} + (c_4 + c_5x)e^{-x} + c_6e^{2x} + c_7e^{5x}$

- **Case (iii)** when roots are **Imaginary and different** complex
- Suppose the auxiliary equation has got two roots $(\alpha + i\beta)$ *and* $(\alpha - i\beta)$ then the part of the solution of the equation corresponding to these roots will be $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

for eg. $3+2i, 3-2i, 1+i, 1-i$

$y_c = e^{3x}(C_1 \cos 2x + c_2 \sin 2x)$
$+ e^{x}(c_3 \cos x + c_4 \sin x)$

- **Case (iv)** when roots are **Imaginary and equal** i.e. repeated
- **(a)** Suppose $(\alpha \pm i\beta)$ occurs twice then the part of the solution with reference to these roots will be $e^{\alpha x}[(C_1 + C_2x)cos\beta x + (C_3 + C_4x)sin\beta x]$

for eg. Suppose $1 \pm i$ repeats twice

$y_c = e^{x}[(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x]$

- **(b)** Suppose $(\alpha \pm i\beta)$ occurs thrice then the part of the solution with reference to these roots will be $e^{\alpha x}[(C_1 + C_2x + C_3x^2)\cos \beta x + (C_4 + C_5x + C_6x^2)\sin \beta x]$

Suppose we get roots of Auxilliary eqn as

$m = 1, 1, -2, 3, 5 \pm i, (2 \pm 3i)$ twice

$$y_c = (c_1 + c_2 x)e^{x} + c_3 e^{-2x} + c_4 e^{3x} + e^{5x}(c_5 \cos x + c_6 \sin x)$$
$$+ e^{2x}\left[(c_7 + c_8 x)\cos 3x + (c_9 + c_{10} x)\sin 3x\right]$$

- **(5)** When the r.h.s $X = 0$ then complete solution = complementary function (i.e no need to find particular integral)
- **(6)** When the r.h.s $X \neq 0$ then we find Particular Integral using following rules.
- $P.I = \frac{1}{f(D)} X$
- The method of finding Particular Integral depends upon the nature of the right hand side X.

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$EXAMPLE - 1: \frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Soln:- $D^3y - 6D^2y + 11Dy - 6y = 0$

$(D^3 - 6D^2 + 11D - 6)y = 0$

$f(D)y = 0$

Auxillary eqn $f(m) = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$m = 1, 2, 3$

$\therefore$ Roots are distinct and real

$\therefore$ The solution $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

EXAMPLE-2: $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$

Soln:- $D^3y - 5D^2y + 8Dy - 4y = 0$

$$(D^3 - 5D^2 + 8D - 4)\, y = 0$$
$$f(D)\, y = 0$$

Auxillary Eqn (A.E.) is $f(m) = 0$

$$m^3 - 5m^2 + 8m - 4 = 0$$
$$m = 1, 2, 2$$

The roots are real and repeated

soln is $y = c_1 e^{x} + (c_2 + c_3 x)\, e^{2x}$

Ex 3:- $\dfrac{d^4 y}{dx^4} + k^4 y = 0$

Soln:- $D^4 y + k^4 y = 0$

$$(D^4 + k^4)\, y = 0$$

A.E is $m^4 + k^4 = 0$

$$m^4 + 2m^2k^2 + k^4 - 2m^2k^2 = 0$$
$$(m^2 + k^2)^2 - (\sqrt{2}mk)^2 = 0$$
$$(m^2 + k^2 + \sqrt{2}mk)(m^2 + k^2 - \sqrt{2}mk) = 0$$
$$m^2 + \sqrt{2}mk + k^2 = 0$$
$$m = \frac{-\sqrt{2}k \pm \sqrt{2k^2 - 4k^2}}{2}$$
$$m = \frac{k}{\sqrt{2}}(-1 \pm i)$$

Similarly, $m^2 - \sqrt{2}mk + k^2 = 0$

$$m = \frac{k}{\sqrt{2}}(1 \pm i)$$

$$m^4 + k^4 = 0$$
$$m^4 = -k^4$$
$$m = k(-1)^{1/4}$$
$$= k\left[\cos\pi + i\sin\pi\right]^{1/4}$$
$$= k\left[\cos\left(\frac{2k+1}{4}\right)\pi + i\sin\left(\frac{2k+1}{4}\right)\pi\right]$$
$$k = 0, 1, 2, 3$$

we have complex, distinct roots

$$m = \frac{k}{\sqrt{2}}(1 \pm i), \quad \frac{k}{\sqrt{2}}(-1 \pm i)$$

$\therefore$ The soln is

$$y = e^{\frac{k}{\sqrt{2}}x}\left(c_1 \cos \frac{k}{\sqrt{2}}x + c_2 \sin \frac{k}{\sqrt{2}}x\right)$$

$$+ e^{-\frac{k}{\sqrt{2}}x}\left(c_3 \cos \frac{k}{\sqrt{2}}x + c_4 \sin \frac{k}{\sqrt{2}}x\right)$$

Ex 4 :- $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

Soln :-

$$D^4y + 2D^2y + y = 0$$

$$(D^4 + 2D^2 + 1)y = 0$$

A.E is $m^4 + 2m^2 + 1 = 0$

$$(m^2+1)^2 = 0$$

$m^2 + 1 = 0$ twice

$m^2 = -1$ twice

$m = \pm i$ twice

roots are complex and repeated

$m = 0 \pm i$ twice

$$y = e^{0x}\left[(C_1 + C_2 x)\sin x + (C_3 + C_4 x)\cos x\right]$$

Ex 5 $\{(D^2+1)^3(D^2+D+1)^2\}y = 0$

Soln :- $\{(D^2+1)^3(D^2+D+1)^2\}y = 0$

A.E. $(m^2+1)^3(m^2+m+1)^2 = 0$

$(m^2+1)^3 = 0 \Rightarrow m^2+1=0$ 3 times

$\Rightarrow m^2 = -1$ 3 times

$\Rightarrow m = \pm i$ 3 times

Also $(m^2+m+1)^2 = 0 \Rightarrow m^2+m+1 = 0$ 2 times

$\Rightarrow m = \dfrac{-1 \pm i\sqrt{3}}{2}$ 2 times

$\therefore$ Roots are complex and repeated

$m = \pm i$ 3 times, $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ 2 times.

$\therefore$ soln is

$$y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$$

$$+ e^{-\frac{1}{2}x}\left[(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right]$$

Ex:- $\{(D-1)^4 (D^2+2D+2)^2\} y = 0$

AE is $(m-1)^4 (m^2+2m+2)^2 = 0$

$(m-1)^4 = 0 \Rightarrow m-1 = 0$ 4 times

$= m = 1$ repeated 4 times

$(m^2+2m+2)^2 = 0 \Rightarrow m^2+2m+2 = 0$ 2 times

$\Rightarrow m = -1 \pm i$ 2 times

$\therefore$ soln is

$$y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) e^x$$

$$+ e^{-x}\left[(c_5 + c_6 x)\cos x + (c_7 + c_8 x)\sin x\right]$$

Ex:- $(D^2-2D-4)y=0$.

A.E. $m^2-2m-4=0$

$$m=\frac{2\pm\sqrt{4+16}}{2}=1\pm\sqrt{5}$$

$$y=c_1e^{(1+\sqrt{5})x}+c_2e^{(1-\sqrt{5})x}$$

$$=e^{x}\left[A\cosh\sqrt{5}x+B\sinh\sqrt{5}x\right]$$

Ex:- $\frac{d^4y}{dx^4}+4\frac{d^3y}{dx^3}+8\frac{d^2y}{dx^2}+8\frac{dy}{dx}+4y=0$

Soln:- A.E. $m^4+4m^3+8m^2+8m+4=0$

$m=-1\pm i$ repeated 2 times

$$y=e^{-x}\left[(c_1+c_2x)\cos x+(c_3+c_4x)\sin x\right]$$