SIMPLE APPLICATIONS OF DIFFERENTIAL EQUATION OF FIRST ORDER AND FIRST DEGREE TO ELECTRICAL AND MECHANICAL ENGINEERING PROBLEM

1. Introduction

In Chapter 1, we have learnt some techniques of solving differential equations of certain types. We have also remarked that the differential equations arise in the study of some engineering, science and social problems. However, formulating a differential equation in a particular situation needs detailed study of the subject to which the problem belongs. We shall remain satisfied here with solving differential equations related to electrical and mechanical problems when the equations are given.

2. Applications of Differential Equations of First Order and First Degree (Mechanical Engineering)

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : An equation in the theory of stability of an aeroplane is $\frac{dv}{dt} = g\cos\alpha - kv$, v being velocity and g, k being constants. It is observed that at time $t = 0$, the velocity $v = 0$. Solve the equation completely.

Sol. : The given differential equation is of variable separable type.

$$\int \frac{dv}{g\cos\alpha - kv} = \int dt + c \qquad \therefore -\frac{1}{k}\log(g\cos\alpha - kv) = t + c$$

By data when $t = 0$, $v = 0$ $\qquad \therefore -\frac{1}{k}\log g\cos\alpha = c$

$$\therefore t = \frac{1}{k}\log g\cos\alpha - \frac{1}{k}\log(g\cos\alpha - kv) = \frac{1}{k}\log\left(\frac{g\cos\alpha}{g\cos\alpha - kv}\right)$$

$$\therefore \frac{g\cos\alpha}{g\cos\alpha - kv} = e^{kt} \qquad \therefore \frac{g\cos\alpha - kv}{g\cos\alpha} = e^{-kt}$$

$$\therefore v = \frac{g\cos\alpha}{k}(1 - e^{-kt}).$$

Example 2 (b) : The equation of motion of a body falling under gravity is given by $\frac{dv}{dt} = g - kv^2$. Find the velocity and distance travelled as a function of time. Given $v = 0$ at $t = 0$.

Sol. : We have $\frac{dv}{dt} = g - kv^2$. $\qquad$ Now taking $k = \frac{g}{\lambda^2}$ for convenience

$$\frac{dv}{dt} = g - \frac{g}{\lambda^2}v^2 = \frac{g}{\lambda^2}(\lambda^2 - v^2)$$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{\lambda^2 - v^2} = \frac{g}{\lambda^2}dt$$

By integration, $\frac{1}{2\lambda}\log\left(\frac{\lambda + v}{\lambda - v}\right) = \frac{g}{\lambda^2}t + c \qquad \therefore \quad \frac{1}{\lambda}\cdot\frac{1}{2}\log\left(\frac{1 + v/\lambda}{1 - v/\lambda}\right) = \frac{g}{\lambda^2}t + c$

Since $\tan h^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), \qquad \frac{1}{\lambda}\tan h^{-1}\frac{v}{\lambda} = \frac{g}{\lambda^2}t + c$

But by data, when $t = 0$, $v = 0 \quad \therefore \quad c = 0$

$$\therefore \quad \frac{1}{\lambda}\tan h^{-1}\frac{v}{\lambda} = \frac{g}{\lambda^2}t \qquad \therefore \tan h^{-1}\frac{v}{\lambda} = \frac{gt}{\lambda}$$

$$\therefore \quad v = \lambda \tan h\left(\frac{gt}{\lambda}\right) \qquad \text{.................. (1)}$$

But $v = \frac{dx}{dt}, \quad \therefore \quad \frac{dx}{dt} = \lambda \tan h\left(\frac{gt}{\lambda}\right) = \lambda\frac{\sin h(gt/\lambda)}{\cos h(gt/\lambda)}$

By integration, $x = \lambda \cdot \frac{\lambda}{g}\log\cos h\left(\frac{gt}{\lambda}\right) + c.$

But by data, when $t = 0$, $x = 0 \quad \therefore \quad c = 0$

$$\therefore \quad x = \frac{\lambda^2}{g}\log\cos h\left(\frac{gt}{\lambda}\right) \qquad \text{............... (2)}$$

Thus, the velocity of the body is given by (1) and the distance travelled is given by (2).

Example 3 (b) : In the above example show further that the velocity of the body approaches a limiting value as $t \to \infty$.

Sol. : As proved above

$$v = \lambda \tan h\left(\frac{gt}{\lambda}\right) = \lambda\left(\frac{e^{gt/\lambda} - e^{-gt/\lambda}}{e^{gt/\lambda} + e^{-gt/\lambda}}\right)$$

$$\therefore \quad v = \lambda\left(\frac{1 - e^{-2gt/\lambda}}{1 + e^{-2gt/\lambda}}\right) \qquad \therefore \quad \lim_{t\to\infty} v = \lim_{t\to\infty} \lambda\left(\frac{1 - e^{-2gt/\lambda}}{1 + e^{-2gt/\lambda}}\right)$$

As $t \to \infty$, $e^{-2gt/\lambda} \to 0 \quad \therefore \quad \lim_{t\to\infty} v = \lambda$. If as $t \to \infty$, $v \to v_0$, then we have

$$v_0 = \lambda = \sqrt{\frac{g}{k}}.$$

Example 4 (b) : The distance x descended by a parashuter satisfied the differential equation $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right)$ where v is the velocity, k, g are constants. If $v = 0$ and $x = 0$ at $t = 0$, show that $x = \frac{k^2}{g}\log\cos h\left(\frac{gt}{k}\right)$.

Sol. : We have $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right) = \frac{g}{k^2}(k^2 - v^2)$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{k^2 - v^2} = \frac{g}{k^2}\,dt$$

Now proceeding as in Example 2, we get

$$x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right).$$

Example 5 (b) : The differential equation of a body of mass m falling from rest subjected to the force of gravity and an air resistance proportional to the square of the velocity is given by $mv\dfrac{dv}{dx} = ka^2 - kv^2$. If it falls through a distance x and possesses a velocity v at that instant, prove that $\dfrac{2kx}{m} = \log\left(\dfrac{a^2}{a^2 - v^2}\right)$ where $mg = ka^2$.

Sol. : We have $\quad mv\dfrac{dv}{dx} = ka^2 - kv^2 = k(a^2 - v^2)$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{v}{a^2 - v^2}\,dv = \frac{k}{m}\,dx \qquad \text{.................... (1)}$$

By integration, $\quad -\dfrac{1}{2}\log(a^2 - v^2) = \dfrac{k}{m}x + \log c$

But by data when $t = 0$, $x = 0$, $v = 0$.

$$\therefore \quad \log c = -\frac{1}{2}\log a^2 \qquad \therefore \quad -\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} - \frac{1}{2}\log a^2$$

$$\therefore \quad \frac{2kx}{m} = \log a^2 - \log(a^2 - v^2) \qquad \therefore \quad \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right).$$

Note ....

Compare this example with the solved Ex. No. 2 above.

Example 6 (b) : The differential equation of a moving body opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2 where x and v are the displacement and velocity of the particle at that time is given by $v\dfrac{dv}{dx} = -cx - bv^2$. Find the velocity of the particle in terms of x, if it starts from rest.

Sol. : We have $\quad v\dfrac{dv}{dx} = -cx - bv^2$. $\qquad$ Putting $v^2 = y$, $v\dfrac{dv}{dx} = \dfrac{1}{2}\dfrac{dy}{dx}$

$$\therefore \quad \frac{1}{2}\cdot\frac{dy}{dx} + by = -cx \qquad \therefore \quad \frac{dy}{dx} + 2by = -2cx$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int 2b\,dx} = e^{2bx}$$

$\therefore$ The solution is $\; y\,e^{2bx} = \displaystyle\int e^{2bx}(-2cx)\,dx + c'$

$$\therefore \quad y\,e^{2bx} = -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int\frac{e^{2bx}}{2b}\cdot(1)\cdot dx\right] + c' \qquad \text{[Integrate by parts]}$$

$$\therefore \quad y\,e^{2bx} = -2c\left[x\cdot\frac{e^{2bx}}{2b} - \frac{e^{2bx}}{4b^2}\right] + c'.$$

Resubstituting $y = v^2$,

$$v^2 e^{2bx} = -\frac{cx}{b}\cdot e^{2bx} + \frac{c}{2b^2}e^{2bx} + c' \qquad \text{.................... (1)}$$

By data, when $x = 0$, $v = 0$ $\quad \therefore c' = -\dfrac{c}{2b^2}$

$$\therefore \quad v^2 e^{2bx} = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} - \frac{c}{2b^2}$$

$$\therefore \quad v^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2}e^{-2bx} = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}.$$

Example 7 (b) : The distance x descended by a parashuter satisfies the differential equation

$$\left(\frac{dx}{dt}\right)^2 = k^2[1 - e^{-2gx/k^2}]$$

where k and g are constants. If $x = 0$ when $t = 0$, show that $x = \dfrac{k^2}{g}\log\cos h\left(\dfrac{gt}{k}\right)$.

Sol. : We have $\dfrac{dx}{dt} = k\sqrt{1 - e^{-2gx/k^2}}$ $\quad \therefore \quad \dfrac{dx}{\sqrt{1 - e^{-2gx/k^2}}} = k\,dt$ .... (1)

Let $\sqrt{1 - e^{-2gx/k^2}} = u$ $\quad \therefore 1 - e^{-2gx/k^2} = u^2$

$$\therefore \quad e^{-2gx/k^2}\cdot\frac{g}{k^2}\cdot dx = u\,du \quad \therefore \quad (1 - u^2)\frac{g}{k^2}dx = u\,du$$

$$\therefore \quad dx = \frac{k^2}{g}\cdot\frac{u}{1 - u^2}du$$

Hence, from (1), we get,

$$\frac{k^2}{g}\cdot\frac{u}{1 - u^2}\cdot\frac{1}{u}du = k\,dt \quad \therefore \quad \frac{k}{g}\cdot\frac{du}{1 - u^2} = dt + c$$

By integration, $\quad \dfrac{k}{g}\cdot\dfrac{1}{2}\log\left(\dfrac{1+u}{1-u}\right) = t + c$

But $\dfrac{1}{2}\log\left(\dfrac{1+u}{1-u}\right) = \tan h^{-1}u$ $\quad \therefore \quad \dfrac{k}{g}\tan h^{-1}u = t + c$

But by data when $t = 0$, $x = 0$ and hence $u = 0$ $\therefore c = 0$

$$\therefore \quad \frac{k}{g}\tan h^{-1}u = t \quad \therefore \tan h^{-1}u = \frac{gt}{k}$$

$$\therefore \quad u = \tan h\frac{gt}{k} \quad \therefore u^2 = \tan h^2\left(\frac{gt}{k}\right)$$

$$\therefore \quad 1 - e^{-2gx/k^2} = \tan h^2\left(\frac{gt}{k}\right)$$

$$\therefore \quad e^{-2gx/k^2} = 1 - \tan h^2\left(\frac{gt}{k}\right) = \sec h^2\left(\frac{gt}{k}\right)$$

$\therefore \quad e^{2gx/k^2} = \cos h^2\left(\frac{gt}{k}\right) \qquad \therefore \quad \frac{2gx}{k^2} = 2\log\cos h\left(\frac{gt}{k}\right)$

$\therefore \quad x = \frac{k^2}{g}\log\cos h\left(\frac{gt}{k}\right).$

Example 8 (b) : The differential equation of a particle moving in a straight line with acceleration $k\left(x+\frac{a^4}{x^3}\right)$ directed towards origin is $v\frac{dv}{dx} = -k\left(x+\frac{a^4}{x^3}\right)$. If it starts from rest at a distance a from the origin, prove that it will arrive at the origin at the end of time $\frac{\pi}{4\sqrt{k}}$.

Sol. : We have $v\frac{dv}{dx} = -k\left(x+\frac{a^4}{x^3}\right)$

(Negative sign because the acceleration is directed towards the origin.)

This is a differential equation of variable separable type.

$$\therefore \quad \int v\,dv = -k\int\left(x+\frac{a^4}{x^3}\right)dx$$

$$\therefore \quad \frac{v^2}{2} = -k\left(\frac{x^2}{2}-\frac{a^4}{2x^2}\right)+c = -k\left(\frac{x^4-a^4}{2x^2}\right)+c$$

By data when $x = 0$, $v = 0 \quad \therefore c = 0$

$$\therefore \quad v^2 = k\left(\frac{a^4-x^4}{x^2}\right) \qquad \therefore \quad v = \pm\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x} \qquad \therefore \quad v = -\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x}.$$

We take the negative sign, since acceleration is directed towards origin.

$$v = \frac{dx}{dt} = -\sqrt{k}\cdot\frac{\sqrt{a^4-x^4}}{x} \qquad \therefore \quad \int\frac{x\,dx}{\sqrt{a^4-x^4}} = -\sqrt{k}\int dt + c$$

Putting $x^2 = u$, $x\,dx = \frac{1}{2}du$,

$$\frac{1}{2}\cdot\int\frac{du}{\sqrt{(a^2)^2-u^2}}du = -\sqrt{k}\int dt + c$$

$$\therefore \quad \frac{1}{2}\sin^{-1}\left(\frac{u}{a^2}\right) = -\sqrt{k}\cdot t + c \qquad \therefore \quad \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) = -\sqrt{k}\cdot t + c$$

By data when $t = 0$, $x = a \quad \therefore c = \frac{\pi}{4}$

$$\therefore \quad \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) = -\sqrt{k}\cdot t + \frac{\pi}{4}$$

The particle will arrive at the origin where $x = 0$.

$$\therefore \quad 0 = -\sqrt{k}\cdot t + \frac{\pi}{4} \qquad \therefore \quad t = \frac{\pi}{4\sqrt{k}}.$$

Example 9 (b) : The differential equation of a body fired vertically from the earth, if it is acted upon by gravitational force only is given by $v\dfrac{dv}{dx} = -\dfrac{gr^2}{x^2}$. Find the initial velocity of a body supposed to escape. (r is the radius of the earth and x is the distance of the body from the earth.)

Sol. : We have $v\dfrac{dv}{dx} = -\dfrac{gr^2}{x^2}$.

This is a differential equation of variable separable type.

$$\therefore \quad v\,dv = -gr^2 \cdot \frac{dx}{x^2}$$

By integration, $\dfrac{v^2}{2} = \dfrac{gr^2}{x} + c$

If u is the required velocity on the surface of the earth where $x = r$ then

$$\frac{u^2}{2} = gr + c \qquad \therefore \quad c = \frac{u^2}{2} - gr$$

$$\therefore \quad \frac{v^2}{2} = \frac{gr^2}{x} + \frac{u^2}{2} - gr \qquad \therefore \quad v^2 = \frac{2gr^2}{x} + u^2 - 2gr$$

This is the equation of motion of a body projected from the surface of the earth with initial velocity u.

If the body is not to return to the earth its velocity v must be always positive. (If the velocity v becomes zero the body will come to rest and then will start to descend.) As x increases, $2gr^2/x$ decreases. Hence, v will be positive if

$$u^2 - 2gr \geq 0 \qquad \therefore \quad u^2 \geq 2gr \qquad i.e. \quad u \geq \sqrt{2gr}.$$

$\therefore$ The **least** velocity of projection $= \sqrt{2gr}$

A particle projected with this velocity will never return to the earth. This is called the **escape velocity** from the earth.

Note ....

Taking the radius of the earth $r = 3960$ miles and $g = 32.17$ ft /sec^2, the escape velocity comes out to be **7 miles per sec.**

Example 10 (b) : The differential equation of a body of mass m projected vertically upwards with velocity V with air resistance k times the velocity is given by $\dfrac{dv}{dt} = -g - \dfrac{kv}{m}$. Show that the particle will reach maximum height in time $\dfrac{m}{k}\log\left(1 + \dfrac{kV}{mg}\right)$.

Sol. : We have $\dfrac{dv}{dt} = -g - \dfrac{kv}{m}$

This is a differential equation of variable separable type.

$$\therefore \quad \frac{dv}{g + (k/m)v} = -dt \qquad \text{.....(1)}$$

By integration, $\dfrac{m}{k}\log\left(g + \dfrac{k}{m}v\right) = -t + c$

When $t = 0$, $v = V$, $\frac{m}{k}\log\left(g + \frac{k}{m}V\right) = c$

$$\therefore\ t = \frac{m}{k}\log\left(g + \frac{k}{m}V\right) - \frac{m}{k}\log\left(g + \frac{k}{m}v\right) = \frac{m}{k}\log\left(\frac{g + (k/m)V}{g + (k/m)v}\right)$$

When the body attains maximum height, $v = 0$.

$$\therefore\ t = \frac{m}{k}\log\left(\frac{g + (k/m)V}{g}\right) = \frac{m}{k}\log\left(1 + \frac{k}{mg}V\right).$$

Example 11 (b) : The differential equation of a body projected vertically upwards in air, considering air resistance, is given by $\frac{dv}{dt} = -g - kv$. Show that the distance travelled by the particle at any time t is given by $x = \left(\frac{g}{k^2} + \frac{u}{k}\right)(1 - e^{-kt}) - \frac{g}{k}.t$ where u is the initial velocity.

Sol. : We have $\frac{dv}{dt} = -g - kv$. $\therefore\ \frac{dv}{g + kv} = -dt$

By integration, we get $\frac{1}{k}\log(g + kv) = -t + c$

Initially when $t = 0$, $v = u$. $\therefore\ \frac{1}{k}\log(g + ku) = c$

$$\therefore\ \frac{1}{k}\log(g + kv) = -t + \frac{1}{k}\log(g + ku) \qquad \therefore\ t = \frac{1}{k}\log\left(\frac{g + ku}{g + kv}\right)$$

$$\therefore\ \frac{g + ku}{g + kv} = e^{kt} \qquad \therefore\ (g + ku)e^{-kt} = g + kv$$

$$\therefore\ v = -\frac{g}{k} + \left(\frac{g + ku}{k}\right)e^{-kt} \qquad \therefore\ \frac{dx}{dt} = -\frac{g}{k} + \left(\frac{g + ku}{k}\right)e^{-kt}$$

By integration, we get $x = -\frac{g}{k}t - \left(\frac{g + ku}{k^2}\right)e^{-kt} + c$

Initially when $t = 0$, $x = 0$ $\therefore\ c = \frac{g + ku}{k^2}$

$$\therefore\ x = -\frac{g}{k}t - \left(\frac{g + ku}{k^2}\right)e^{-kt} + \left(\frac{g + ku}{k^2}\right)$$

$$= \left(\frac{g + ku}{k^2}\right)(1 - e^{-kt}) - \frac{g}{k}t$$

$$\therefore\ x = \left(\frac{g}{k^2} + \frac{u}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t.$$

EXERCISE - I

Solve the following examples : Class (b) : 6 Marks

1. The differential equation of a body in motion is given by $\frac{dv}{dt} = k\left(1 - \frac{t}{T}\right)$ where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained. (At $t = 0$, $s = 0$, $v = 0$). [**Ans. :** $t = T$, $v = \frac{kT}{2}$, $s = \frac{kT^2}{2}$]

2. The velocity of a bullet fired in a sand tank is given by $\frac{dv}{dt} = -k\sqrt{v}$. If at $t = 0$, $v = V$, find how long it will take to come to rest.

[Ans. : $2\sqrt{V}/k$]

3. A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{x/3}$.

4. The equation of motion of a particle moving in a straight line is given by $v\frac{dv}{dx} = -\frac{k}{x^3}$. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\frac{a}{2}$ from the origin at $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$.

5. The equation of motion of a particle moving in a straight line in a resisting medium is given by $\frac{dv}{ds} = -kv^2$. If u is the initial velocity, prove that $v = \frac{u}{1 + ksu}$ and $t = \frac{s}{u} + \frac{ks^2}{2}$.

6. The differential equation of a body falling from rest subjected to the force of gravity and air resistance is given by $v\frac{dv}{dx} + \frac{n^2}{g}v^2 = g$. Prove that the velocity is given by $v^2 = \frac{g^2}{n^2}(1 - e^{-2n^2x/g})$. ($v = 0$ at $x = 0$)

3. Applications of Differential Equations of First Order and First Degree (Electrical Engineering)

We shall consider simple electrical circuits containing resistance (R), inductance (L), capacitance (C) and voltage (V) or electromotive force (E).

We denote the charge by q and the rate of flow of electricity *i.e.* current by i. With these notations, we know that

1. $i = \frac{dq}{dt}$ or $q = \int i\,dt$
2. Voltage drop across resistance $R = R_i$
3. Voltage drop across inductance $L = L\frac{di}{dt}$
4. Voltage drop across capacitance $C = \frac{q}{C}$.

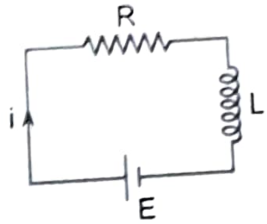
The differential equation of an electrical circuit can be formed by using the following two Kirchhoff's Laws.

1. The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.
2. The algebraic sum of the currents flowing into or from any terminal of the circuit is zero.

We shall consider the following cases.

(1) **R-L-E circuit**

Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. Let i be the current in the circuit at any time t. Then by Kirchhoff's law.



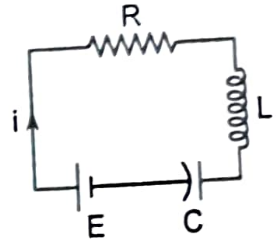
Sum of voltage drops across R and $L = E$.

$$\therefore \quad Ri + L\frac{di}{dt} = E. \qquad \therefore \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}.$$

This is the required differential equation of the circuit.

(2) **R-L-C-E circuit**

Consider a circuit containing resistance R, inductance L and capacitance C in series with a voltage source (battery) E. Let i be the current at any time t. Then by Kirchhoff's Law.



Sum of voltage drops across R, L, $C = E$.

$$\therefore \quad Ri + L\frac{di}{dt} + \frac{q}{C} = E \qquad \therefore \quad \frac{di}{dt} + \frac{R}{L}i + \frac{q}{CL} = \frac{E}{L}$$

Putting $i = \dfrac{dq}{dt}$, we get

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{CL} = \frac{E}{L}.$$

We need the following two integrals often in solving the differential equations of electrical circuits

1. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\frac{a}{\sqrt{a^2+b^2}}\sin bx - \frac{b}{\sqrt{a^2+b^2}}\cos bx\right) \quad \text{.................... (1)}$$

Putting $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\Phi$, $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\Phi$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx - \Phi) \quad \text{.................. (1A)}$$

2. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left(\frac{a}{\sqrt{a^2+b^2}}\cos bx + \frac{b}{\sqrt{a^2+b^2}}\sin bx\right) \quad \text{.................... (2)}$$

Putting $\dfrac{a}{\sqrt{a^2+b^2}} = \cos\Phi$, $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\Phi$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx - \Phi). \quad \text{.................. (2A)}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt}+Ri=E$. Find the current i at time t if at $t=0$, $i=0$ and L, R, E are constants.

(M.U. 1995, 2009, 12, 16, 18)

Sol. : The given equation $\frac{di}{dt}+\frac{Ri}{L}=\frac{E}{L}$ is linear of the type $\frac{dy}{dx}+Py=Q$.

$\therefore$ Its solution is $i\cdot e^{\int(R/L)dt}=\int e^{\int(R/L)dt}\cdot\frac{E}{L}\cdot dt+c$

$\therefore\ i\cdot e^{Rt/L}=\frac{E}{L}\int e^{Rt/L}dt+c=\frac{E}{L}\cdot e^{Rt/L}\cdot\frac{L}{R}+c=\frac{E}{R}e^{Rt/L}+c$

When $t=0,\ i=0\ \therefore c=-\frac{E}{R}$

$\therefore\ i\cdot e^{Rt/L}=\frac{E}{R}e^{Rt/L}-\frac{E}{R}=\frac{E}{R}(e^{Rt/L}-1)$

$\therefore\ i=\frac{E}{R}(1-e^{-Rt/L})$ .... (3)

Example 2 (b) : A constant e.m.f. E volts is applied to a circuit containing a constant resistance R ohms. in series and a constant inductance L henries. The current i at any time t is given by $L\frac{di}{dt}+Ri=E$. If the initial current is zero, show that the current builds up to half its theoretical maximum value in $\frac{L}{R}\cdot\log 2$ seconds.

(M.U. 2000)

Sol. : As in example 1 the current is given by $i=\frac{E}{R}\left(1-e^{-Rt/L}\right)$

As $t\to\infty$, $e^{-Rt/L}\to 0$ and the current reaches its theoretical maximum value say I.

$\therefore\ I=\frac{E}{R}$

When $i=\frac{I}{2}$, we get $\frac{I}{2}=I\left(1-e^{-Rt/L}\right)\quad\therefore\ \frac{1}{2}=1-e^{-Rt/L}$

$\therefore\ e^{-Rt/L}=\frac{1}{2}\quad\therefore\ e^{Rt/L}=2$

$\therefore\ \frac{Rt}{L}=\log 2\quad\therefore\ t=\frac{L}{R}\log 2.$ .... (4)

Example 3 (b) : A resistance of 100 ohms and inductance 0.5 henries are connected in series with a battery of 20 volts. Find the current at any instant if the relation between L, R and E is

$$L\frac{di}{dt}+Ri=E.$$

(M.U. 2015, 18)

Sol. : The given equation can be written as

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}\qquad\text{.................... (i)}$$

It is linear of the type $\frac{dy}{dx}+Py=Q$.

Putting the given values of $R = 100$, $L = 0.5$ and $E = 20$ in (i), we get

$$\frac{di}{dt} + \frac{100}{0\cdot5}i = \frac{20}{0\cdot5} \quad i.e., \quad \frac{di}{dt} + 200\,i = 40$$

Now, $\int P\,dx = \int 200\,dt = 200\,t$

Hence, the solution is

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + c$$

$$\therefore \quad i \cdot e^{200\,t} = \int 40 \cdot e^{200\,t}\,dt + c = 40 \cdot \frac{e^{200\,t}}{200} + c = \frac{1}{50} \cdot e^{200\,t} + c$$

When $t = 0$, $i = 0$ then $\frac{1}{50}e^0 + c = 0 \quad \therefore \quad c = -\frac{1}{50}$.

$$\therefore \quad i \cdot e^{200\,t} = \frac{1}{50} \cdot e^{200\,t} - \frac{1}{50}$$

$$\therefore \quad i = \frac{1}{50} - \frac{1}{50}e^{-200\,t} = \frac{1}{50}(1 - e^{-200\,t}) = 0\cdot02\,(1 - e^{-200\,t})$$

Example 4 (b) : The differential equation of a circuit with inductance L and resistance R is given by $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$. Show that the current at any time t is given by $i = \frac{E}{R - aL}(e^{-at} - e^{-Rt/L})$.

(Given $i = 0$ at $t = 0$) **(M.U. 1998, 2010)**

Sol. : We have $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$.

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$. Its solution is

$$i \cdot e^{\int (R/L)\,dt} = \int e^{\int (R/L)\,dt} \cdot \frac{E}{L}e^{-at}\,dt + c$$

$$\therefore \quad i \cdot e^{Rt/L} = \frac{E}{L}\int e^{(Rt/L)} \cdot e^{-at}\,dt + c = \frac{E}{L}\int e^{[(R/L)-a]t}\,dt + c$$

$$i \cdot e^{Rt/L} = \frac{E}{L} \cdot \frac{e^{[(R/L)-a]t}}{(R/L) - a} + c = \frac{E}{R - La} \cdot e^{[(R/L)-a]t} + c$$

By data, when $t = 0$, $i = 0$, $\quad \therefore c = -\frac{E}{R - La}$

$$\therefore \quad i \cdot e^{Rt/L} = \frac{E}{R - La} \cdot e^{[(R/L)-a]t} - \frac{E}{R - La}$$

$$\therefore \quad i = \frac{E}{R - La} \cdot e^{-at} - \frac{E}{R - La} \cdot e^{-Rt/L} = \frac{E}{R - La}(e^{-at} - e^{-Rt/L}).$$

Example 5 (b) : The current in a circuit containing an inductance L, resistance R, and voltage $E \sin \omega t$ is given by $L\frac{di}{dt} + Ri = E \sin \omega t$. If $i = 0$ at $t = 0$, show that

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}}\left[\sin(\omega t - \Phi)\,a + e^{-Rt/L}\sin\Phi\right] \text{ where } \Phi = \tan^{-1}\left(\frac{L\omega}{R}\right).$$ **(M.U. 1995, 2013)**

Sol. : The given equation $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin \omega t$ is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$.

Hence, the solution is

$$i \cdot e^{\int (R/L)dt} = \int e^{\int (R/L)dt} \cdot \frac{E}{L} \sin \omega t \, dt + c$$

$$\therefore \quad i \cdot e^{Rt/L} = \int e^{Rt/L} \cdot \frac{E}{L} \sin \omega t \, dt + c$$

Integrating by parts, *i.e.* by (1), page S-9 we get,

$$i \cdot e^{Rt/L} = \frac{E}{L} \cdot \frac{1}{(R^2/L^2) + \omega^2} \cdot e^{Rt/L} \left(\frac{R}{L} \sin \omega t - \omega \cos \omega t \right) + c$$

$$\therefore \quad i \cdot e^{Rt/L} = E \cdot \frac{1}{R^2 + \omega^2 L^2} \cdot e^{Rt/L} (R \sin \omega t - \omega L \cos \omega t) + c$$

When $t = 0$, $i = 0$ $\therefore$ $c = E \cdot \dfrac{\omega L}{R^2 + \omega^2 L^2}$.

$$\therefore \quad i = E \cdot \frac{1}{R^2 + \omega^2 L^2} \cdot (R \sin \omega t - \omega L \cos \omega t) + e^{-Rt/L} \cdot E \cdot \frac{\omega L}{R^2 + \omega^2 L^2}$$

$$= E \cdot \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \cdot \left(\frac{R}{\sqrt{R^2 + \omega^2 L^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \cos \omega t \right)$$

$$+ e^{-Rt/L} \cdot \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}.$$

Putting $\dfrac{R}{\sqrt{R^2 + \omega^2 L^2}} = \cos \Phi$, $\dfrac{L\omega}{\sqrt{R^2 + \omega^2 L^2}} = \sin \Phi$

$$\therefore \quad i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} [\sin (\omega t - \Phi) + e^{-Rt/L} \sin \Phi].$$

Example 6 (b) : The equation of the electromotive force in terms of current i for an electrical circuit having resistance R and a condenser of capacity C in series is $E = Ri + \int \frac{i}{c} dt$. Find the current i at any time t, when $E = E_0 \sin \omega t$.

(M.U. 1991, 94)

Sol. : Putting $E = E_0 \sin \omega t$ in the given equation we have $Ri + \int \frac{i}{c} dt = E_0 \sin \omega t$.

Differentiating it w.r.t. t, we get

$$R \frac{di}{dt} + \frac{i}{c} = E_0 \omega \cos \omega t \quad \therefore \quad \frac{di}{dt} + \frac{i}{Rc} = \frac{E_0 \omega}{R} \cdot \cos \omega t$$

This is a linear differential equation of the type $\dfrac{dy}{dx} + Py = Q$.

Its solution is

$$i \cdot e^{\int (1/Rc)dt} = \int e^{\int (1/Rc)dt} \cdot \frac{E_0 \omega}{R} \cdot \cos \omega t \, dt + k$$

$$\therefore \quad i \cdot e^{t/Rc} = \frac{E_0 \omega}{R} \int e^{t/Rc} \cos \omega t \, dt + k$$

$$\therefore \quad i \cdot e^{t/Rc} = \frac{E_0 \omega}{R} \cdot \frac{e^{t/Rc}}{\sqrt{(1/R^2 c^2) + \omega^2}} \cos (\omega t - \Phi) + k \qquad \text{[By (2A), page S-9]}$$

where $\tan \Phi = \dfrac{\omega}{1/Rc} = Rc\omega$.

$$\therefore \quad i = \frac{E_0 \omega}{R} \cdot \frac{Rc}{\sqrt{1 + R^2c^2\omega^2}} \cos(\omega t - \Phi) + k\,e^{-t/Rc}$$

$$= \frac{E_0 \omega c}{\sqrt{1 + R^2c^2\omega^2}} \cdot \cos(\omega t - \Phi) + k\,e^{-t/Rc}.$$

Example 7 (b) : The charge q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\frac{dq}{dt} + \frac{q}{C} = V$. If $q = 0$ at $t = 0$, show that $q = CV(1 - e^{-t/RC})$. Find also the current flowing into the plate. **(M.U. 1995, 2015)**

Sol. : We are given that $\frac{dq}{dt} + \frac{1}{RC} \cdot q = \frac{V}{R}$.

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$.

Its solution is $q \cdot e^{\int (1/RC)dt} = \int e^{\int (1/RC)dt} \cdot \frac{V}{R} dt + k$

$$\therefore \quad q \cdot e^{t/RC} = \int e^{t/RC} \cdot \frac{V}{R} dt + k = \frac{V}{R} \cdot \frac{e^{t/RC}}{(1/RC)} + k$$

$$= CV \cdot e^{t/RC} + k$$

By data when $t = 0$, $q = 0$ $\therefore$ $k = -CV$

$$\therefore \quad q \cdot e^{t/RC} = CV \cdot e^{t/RC} - CV$$

$$\therefore \quad q = CV - CV e^{-t/RC} = CV(1 - e^{-t/RC})$$

Further $i = \frac{dq}{dt} = CV \cdot e^{-t/RC} \cdot \frac{1}{RC} = \frac{V}{R} \cdot e^{-t/RC}$.

Example 8 (b) : In a circuit of resistance R, self inductance L, the current, i is given by

$$L\frac{di}{dt} + Ri = E\cos pt$$

when E and p are constants, find the current i at time t. **(M.U. 2013)**

Sol. : We have $\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}\cos pt$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$.

$$\therefore \quad \text{I.F.} = e^{\int (R/L)dt} = e^{(R/L)t}$$

$\therefore$ Its solution is

$$i \cdot e^{(R/L)t} = \int e^{(R/L)t} \cdot \frac{E}{L} \cos pt \; dt + c$$

$$= \frac{E}{L} \int e^{(R/L)t} \cdot \cos pt \; dt + c$$

$$= \frac{E}{L}\left[\frac{e^{(R/L)t}}{(R/L)^2 + p^2}\left(\frac{R}{L}\cos pt + p\sin pt\right)\right] + c$$

[See (23), page F-6 in the list of formulae]

$$\therefore \quad i \cdot e^{(R/L)t} = \frac{EL}{R^2 + L^2p^2} \cdot e^{(R/L)t}\left[\frac{R}{L}\cos pt + p\sin pt\right] + c$$

$$\therefore \quad i = \frac{EL}{R^2 + L^2p^2}\left[\frac{R}{L}\cos pt + p\sin pt\right] + c\,e^{-(R/L)t}$$

$$= \frac{E}{R^2 + L^2p^2}[R\cos pt + Lp\sin pt] + c\,e^{-(R/L)t}.$$

EXERCISE - II

Solve the following examples : Class (b) : 6 Marks

1. The current i in a circuit containing a resistance R and a condenser of capacity C farads and connected to a constant e.m.f. E is given by $Ri + \frac{q}{c} = \frac{E}{R}$. Find q, given that $q = 0$ when $t = 0$.

[**Ans. :** $q = EC(1 - e^{-t/CR})$]

2. When the inner of two concentric spheres of radius r_1 and r_2 $(r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at a distance r from the common centre is $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$. Find v in terms or r.

[**Ans. :** $vr = c_1 r - c_2$. Put $\frac{dv}{dr} = z$]

3. When a switch is closed, the current built up in an electric circuit is given by $E = Ri + L\frac{di}{dt}$. If $L = 640$, $R = 250$, $E = 500$ and $i = 0$ when $t = 0$ show that the current will approach 2 amp. when $t \to \infty$.