APPLICATION OF DIFFERENTIAL EQUATIONS

Q.1 A chain coiled up near the edge of a smooth table starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\frac{dv}{dx} + v^2 = gx$. Solve the Differential equation to express v in terms of x.

Solution: The given differential equation is $xv\frac{dv}{dx} + v^2 = gx$ ……….(1)

put $v^2 = y$ $\therefore 2v\frac{dv}{dx} = \frac{dy}{dx}$

$\therefore$ From (1) $x\left(\frac{1}{2}\frac{dy}{dx}\right) + y = gx$

$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right)y = 2g$

It is linear differential equation with $P = \frac{2}{x}$ and $Q = 2g$

$\therefore$ Its solution is $ye^{\int Pdx} = \int Q \cdot e^{\int Pdx}\,dx + c$

$ye^{\int\frac{2}{x}dx} = \int 2g \cdot e^{\int\frac{2}{x}dx}dx + c$

$yx^2 = \int 2g \cdot x^2 dx + c$

$yx^2 = \frac{2}{3}gx^3 + c$

$v^2x^2 = \frac{2g}{3}x^3 + c$

Q.2. In a circuit containing inductance L, resistance R, and voltage E, the current i is given by $L\frac{di}{dt} + Ri = E$. Find the current i at time t if at $t = 0, i = 0$ and L, R, E are constants.

Solution: The given equation $\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is linear of the type $\frac{dy}{dx} + Py = Q$

$\therefore$ Its solution is $i \cdot e^{\int(R/L)dt} = \int \frac{E}{L} \cdot e^{\int(R/L)dt}dt + c$

$\therefore i \cdot e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\,dt + c = \frac{E}{L} \cdot e^{Rt/L} \cdot \frac{L}{R} + c = \frac{E}{R}e^{Rt/L} + c$

When $t = 0, i = 0$ $\therefore c = -\frac{E}{R}$

$\therefore i \cdot e^{Rt/L} = \frac{E}{R}e^{Rt/L} - \frac{E}{R} = \frac{E}{R}\left(e^{Rt/L} - 1\right)$

$\therefore i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

Q.3. In a circuit containing inductance L resistance R and voltage E, the current i is given by $E = Ri + L\frac{di}{dt}$ .If $L = 640h, R = 250\ \Omega\ and\ E = 500$volts and $i = 0$ when $t = 0$, find the time that elapses before the current reaches 90% of its maximum value.

Solution: As in above example the current is given by $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

Maximum value of i is I, can be obtained when $t \to \infty$

$\therefore I = \frac{E}{R}(1 - 0) = \frac{E}{R}$

When $i = 90\%$ if $I = \frac{9}{10}I = \frac{9E}{10R}$

We get $\frac{9}{10}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$

$\frac{9}{10} = 1 - e^{-Rt/L}$

$e^{-Rt/L} = 1 - \frac{9}{10} = \frac{1}{10}$

$e^{Rt/L} = 10$

$Rt/L = \log 10$

$t = \frac{L}{R} \log 10 \text{ sec}$

$\therefore t = \frac{640}{250} \log 10 = 5.89 \text{ sec}$

Q.4. The charge q on the plate of a condenser of capacity C charged through a resistance R by the steady voltage V satisfies the differential equation $R\frac{dq}{dt} + \frac{q}{C} = V$. If $q = 0 \ at \ t = 0$, show that $q = CV(1 - e^{-t/RC})$. Find also the current flowing into the plate.

Solution: We are given that $\frac{dq}{dt} + \frac{1}{Rc} \cdot q = \frac{V}{R}$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$

Its solution is $q \cdot e^{\int (1/Rc)dt} = \int e^{\int (1/Rc)dt} \cdot \frac{V}{R} dt + k$

$\therefore q \cdot e^{t/RC} = \int e^{t/RC} \cdot \frac{V}{R} dt + k = \frac{V}{R} \cdot \frac{e^{t/RC}}{(1/RC)} + k$

By data when $t = 0, q = 0$ $\quad\quad \therefore k = -CV$

$\therefore \ q \cdot e^{t/RC} = CV \cdot e^{t/RC} - CV = e^{t/RC}\left(CV - CVe^{-t/RC}\right)$

$\therefore \ q = CV\left(1 - e^{-t/RC}\right)$

Further $i = \frac{dq}{dt} = CV \cdot e^{-t/RC} \cdot \frac{1}{RC} = \frac{V}{R} \cdot e^{-t/RC}$

Q.5. An equation in the theory of stability of an aeroplane is $\frac{dv}{dt} = g \cos \alpha - kv$, v being velocity and g, k being constants. It is observed that at time $t = 0$, the velocity $v = 0$. Solve the equation completely.