LINEAR DIFFERENTIAL EQUATIONS

Definition: A differential equation is said to be linear if the dependent variable and its derivatives appear only in the first degree. The form of the linear equation of the first order is

$\frac{dy}{dx} + Py = Q$ Where P and Q are function of x or constants only.

For example, $\frac{dy}{dx} + 3xy = x^2$, $\frac{dy}{dx} + y = e^x$ are linear equations.

Method to solve Linear Differential Equations :

(1) First write the equation with the coefficient of $\frac{dy}{dx}$ unity i.e. in the form $\frac{dy}{dx} + Py = Q$

(2) Find $\int Pdx$ and further $I.F = e^{\int P\,dx}$

(3) Multiply the equation by Integrating factor $e^{\int P\,dx}$ it becomes exact and hence can be solved by mere integration.

(4) The solution is $y.\left(e^{\int P\,dx}\right) = \int \left(\left(e^{\int P\,dx}\right).Q\right)\,dx + c$

ANOTHER FORM OF LINEAR DIFFERENTIAL EQUATION :

A differential equation of the form $\frac{dx}{dy} + p'x = Q'$ Where P' and Q' are functions of y only is also a linear differential equation with x and y having interchanged the positions.

Its solution is, $x.\left(e^{\int P'\,dy}\right) = \int \left(\left(e^{\int P'\,dy}\right).Q'\right)dy + c$

EXAMPLES

1. $\frac{dy}{dx} + \left(\frac{1-2x}{x^2}\right)y = 1$

Solution : This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Now, $\int P\,dx = \int \left(\frac{1-2x}{x^2}\right)dx = \int \frac{dx}{x^2} - 2\int \frac{dx}{x} = -\frac{1}{x} - 2\log x$

$\therefore e^{\int Pdx} = e^{-(1/x)-2\log x} = e^{-1/x} \cdot e^{-2\log x} = e^{-1/x} \cdot \frac{1}{x^2}$

$\therefore$ The solution is $ye^{\int Pdx} = \int e^{\int Pdx} \cdot Qdx + c$

$\therefore ye^{-1/x} \cdot \frac{1}{x^2} = \int e^{-1/x} \cdot \frac{1}{x^2} Qdx + c$

$ye^{-1/x} \cdot \frac{1}{x^2} = \int e^{-1/x} \cdot \frac{1}{x^2} dx + c$ $\because \int e^{f(x)} \cdot f'(x)dx = e^{f(x)} + c$

$\therefore$ The solution is $ye^{-1/x} \cdot \frac{1}{x^2} = e^{-1/x} + c$ $\therefore y = x^2 + ce^{1/x} \cdot x^2$

2. $(1 + x + xy^2)dy + (y + y^3)dx = 0$

Solution: We have, $1 + x(1 + y^2) + y(1 + y^2)\frac{dx}{dy} = 0$

$\therefore \frac{dx}{dy} + \frac{x}{y} = -\frac{1}{y(1+y^2)}$

This is a linear differential equation of the form $\frac{dx}{dy} + P'x = Q'$

Now, $\int P'\,dy = \int \frac{dy}{y} = \log y$ $\therefore e^{\int P'dy} = e^{\log y} = y$

$\therefore$ This solution is $x \cdot e^{\int P'dy} = \int e^{\int P'dy} \cdot Q'dy + c$

$\therefore xy = \int y\left[-\frac{1}{y(1+y^2)}\right]dy + c = -\int \frac{dy}{1+y^2} = -\tan^{-1} y + c$

$\therefore xy + \tan^{-1} y = c$

3. $(1 + y^2)dx = \left(e^{tan^{-1}y} - x\right)dy$

Solution: The equation can be written as $\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{e^{\tan^{-1}y}}{1+y^2}$

This is a linear differential equation of the form $\frac{dx}{dy} + P'x = Q'$

Now, $\int P'\,dy = \int \frac{1}{1+y^2} dy = \tan^{-1} y$ $\therefore e^{\int P'dy} = e^{\tan^{-1} y}$

$\therefore$ The solution is $x \cdot e^{\int P' dy} = \int e^{\int P' dy} \cdot Q'\, dy + c$

$\therefore xe^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2} \cdot dy + c$

put $\tan^{-1} y = t \quad \therefore \frac{1}{1+y^2} \cdot dy = dt$

$\therefore xe^{\tan^{-1} y} = \int e^{2t} \cdot dt + c = \frac{1}{2} e^{2t} + c \qquad \therefore xe^{\tan^{-1} y} = \frac{1}{2} e^{2\tan^{-1} y} + c$

EQUATION REDUCIBLE TO LINEAR FORM :

(1) The equation of the type $f'(y)\frac{dy}{dx} + P.f(y) = Q$ Where P and Q are functions of x only can be reduced to linear form as follows.

Let us put $f(y) = v$ then $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ The equation reduces to $\frac{dv}{dx} + Pv = Q$ which is linear.

Its solution is $v.\left(e^{\int P\, dx}\right) = \int \left(\left(e^{\int P\, dx}\right).Q\right) dx + c$

(2) The equation of the type $f'(x)\frac{dx}{dy} + Pf(x) = Q$ Where P and Q are functions of y only can also be reduced to linear form as follows.

Let us put $f(x) = v$ then $f'(x)\frac{dx}{dy} = \frac{dv}{dy}$

$\therefore$ The equation reduces to $\frac{dv}{dy} + Pv = Q$ which is linear.

Its solution is $v.\left(e^{\int P\, dy}\right) = \int \left(\left(e^{\int P\, dy}\right).Q\right) dy + c$

4. $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Solution: Dividing by $\cos^2 y$ the equation can be written as $\sec^2 y \frac{dy}{dx} + \sec^2 y \cdot \sin 2y \cdot x = x^3$ ……..(1)

$\therefore \sec^2 y \frac{dy}{dx} + 2\tan y \cdot x = x^3$

Put $\tan y = v$ and differentiate w.r.t. x, we get $\sec^2 y \frac{dy}{dx} = \frac{dv}{dx}$

Hence, from (1), we get $\frac{dv}{dx} + 2v \cdot x = x^3$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

$\therefore \int P\, dx = \int 2x\, dx = x^2 \qquad \therefore IF = e^{\int Pdx} = e^{x^2}$

$\therefore$ The solution is $ye^{\int Pdx} = \int e^{\int Pdx} \cdot Qdx + c$

$\therefore ve^{x^2} = \int e^{x^2} x^3\, dx + c$

To find the integral on R.H.S. put $x^2 = t$, $\therefore x^2 dx = dt \qquad \therefore xdx = \frac{dt}{2}$

$\therefore \int e^{x^2} x^3\, dx = \int e^t \cdot t \cdot \frac{dt}{2} = \frac{1}{2}[t \cdot e^t - \int e^t \cdot dt] = \frac{1}{2}[te^t - e^t] = \frac{1}{2} e^t (t-1) = \frac{1}{2} e^{x^2}(x^2 - 1)$

$\therefore$ The solution is $ve^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c$

Re sub. $v = tan\, y$

$\therefore \tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2 - 1) + c \qquad \therefore \tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

5. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Solution: The equation can be written as $\frac{dy}{dx} = \frac{e^x}{e^y}(e^x - e^y)$ i.e. $e^y \frac{dy}{dx} + e^y \cdot e^x = e^{2x}$ ……..(1)

Now, put $e^y = v$ and differnetiate w.r.t. x, $\quad e^y \frac{dy}{dx} = \frac{dv}{dx}$

Hence, from (1), we get $\frac{dv}{dx} + e^x \cdot v = e^{2x}$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Its solution is $ve^{\int P dx} = \int e^{\int P dx} \cdot Q\, dx + c$

$\therefore ve^{\int e^x dx} = \int e^{\int e^x dx} \cdot e^{2x}\, dx + c$

$\therefore ve^{e^x} = \int e^{e^x} \cdot e^{2x} \cdot dx + c$

To find the integral on R.H.S. put $e^x = t \quad \therefore e^x dx = dt$

$\therefore \int e^{e^x} e^x \cdot e^x \cdot dx = \int e^t \cdot t\, dt = e^t(t-1)$

$\therefore$ The solution is $ve^{e^x} = e^{e^x}(e^x - 1) + c$

$\therefore v = (e^x - 1) + ce^{-e^x}$

Re sub. $v = e^y$ $\quad \therefore e^y = e^x - 1 + ce^{-e^x}$

6. $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$

Solution: The equation can be written as $e^{2x} + y^2 = y^3 \frac{dx}{dy}$ $\quad \therefore \frac{dx}{dy} - \frac{1}{y} = e^{2x} \cdot \frac{1}{y^3}$

Dividing by e^{-2x}, $\quad e^{-2x}\frac{dx}{dy} - e^{-2x} \cdot \frac{1}{y} = \frac{1}{y^3}$

Putting $e^{-2x} = v$, $\quad \therefore -2e^{-2x}\frac{dx}{dy} = \frac{dv}{dy}$, we get,

$-\frac{1}{2} \cdot \frac{dv}{dy} - \frac{1}{y} \cdot v = \frac{1}{y^3}$ i.e. $\frac{dv}{dy} + \frac{2}{y} \cdot v = -\frac{2}{y^3}$

This is a linear differential equation of the form $\frac{dv}{dy} + Pv = Q$

$\therefore e^{\int P dy} = e^{\int (2/y) dy} = e^{2 \log y} = e^{\log y^2} = y^2$

The solution is $ve^{\int P dy} = \int e^{\int P dy} \cdot Q\, dy + c$

$\therefore v \cdot y^2 = \int y^2 \left(-\frac{2}{y^3}\right) dy + c$

$\therefore vy^2 = \int -\frac{2}{y} dy + c$ $\quad \therefore vy^2 = -2 \log y + c$ $\quad \therefore e^{-2x}y^2 + 2 \log y = c$