INTEGRATING FACTOR :

Some times a given differential equation is not exact but is rendered exact if it is multiplied by a suitable factor, Such a factor is called an **Integrating Factor**

Standard rules of obtaining integrating factors.

Rule 1 : If $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)/N$ is a function of x only, say f(x) then $e^{\int f(x)dx}$ is an integrating factor.

Rule 2 : If $\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)/M$ is a function of y only say f(y) then $e^{-\int f(y)dy}$ is an integrating factor.

Rule 3 : If the equation is of the form $f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$ and $Mx - Ny \neq 0$ then $1/(Mx - Ny)$ is an integrating factor.

Rule 4 : If the equation $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$ then $1/(Mx + Ny)$ is an integrating factor.

EXAMPLES:

1. $(x^2 + y^2 + 1)dx - 2xy\,dy = 0$

Solution: We have, $M = x^2 + y^2 + 1$ and $N = -2xy$ $\therefore \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = -2y$

$\therefore \frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N} = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$

$$\therefore IF = e^{\int -(2/x)dx} = e^{-2\log x} = e^{\log(1/x^2)} = \frac{1}{x^2}$$

Multiplying by the IF, we get, $\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx + \left(-\frac{2y}{x}\right)dy = 0$, which is exact

Now, $\int M\,dx = \int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx = x - \frac{y^2}{x} - \frac{1}{x}$ and

$\int N\,dy = \int(\text{terms in } N \text{ free from } x)\,dy = 0$

$\therefore$ The solution is $x - \frac{y^2}{x} - \frac{1}{x} = c$, i.e. $x^2 - y^2 - 1 = cx$

2. $y(xy + e^x)dx - e^x dy = 0$

Solution: We have, $M = y(xy + e^x)$ and $N = -e^x$ $\therefore \frac{\partial M}{\partial y} = 2xy + e^x$ and $\frac{\partial N}{\partial x} = -e^x$

$\therefore \frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \frac{-e^x-2xy-e^x}{y(xy+e^x)} = \frac{-2(xy+e^x)}{y(xy+e^x)} = \frac{-2}{y} = f(y)$

$\therefore IF = e^{\int(-2/y)dy} = e^{-2\log y} = e^{\log(1/y^2)} = \frac{1}{y^2}$

Multiply the given equation by IF, we get, $\left(x+\frac{e^x}{y}\right)dx - \frac{e^x}{y^2}dy = 0$, which is exact

$\therefore \int M\,dx = \int\left(x+\frac{e^x}{y}\right)dx = \frac{x^2}{2}+\frac{e^x}{y}$

$\int(\text{terms in } N \text{ free from } x)\,dy = 0$

$\therefore$ The solution is $\frac{x^2}{2}+\frac{e^x}{y} = c$

3. $\frac{dy}{dx} = -\frac{x^2y^3+2y}{2x-2x^3y^2}$

Solution: The equation can be written as $(x^2y^3 + 2y)dx + (2x - 2x^3y^2)dy = 0$

i.e. $y(2 + x^2y^2)dx + x(2 - 2x^2y^2)dy = 0$

We have, $M = y(2 + x^2y^2)$ and $N = x(2 - 2x^2y^2)$

$\therefore \frac{\partial M}{\partial y} = 2 + 3x^2y^2$ and $\frac{\partial N}{\partial x} = 2 - 6x^2y^2$ The DE is not exact.

$Mx - Ny = 2xy + x^3y^3 - 2xy + 2x^3y^3 = 3x^3y^3 \neq 0$

$\therefore IF = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$

Multiply the given equation by IF, we get, $\left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \left(\frac{2}{3x^2y^3} - \frac{2}{3y}\right)dy = 0$

Now, $\int M\,dx = \int \left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx = \frac{1}{3}\log x - \frac{1}{3x^2y^2}$

And $\int(\text{terms in } N \text{ free from } x)\,dy = \int -\frac{2}{3}y\,dy = -\frac{2}{3}\log y$

$\therefore$ The solution is $\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$

$\therefore \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c$

4. $\left[2x\, sin\, h\left(\frac{y}{x}\right) + 3ycos\, h\left(\frac{y}{x}\right)\right]dx - 3x.cos\, h\left(\frac{y}{x}\right).dy = 0$

Solution: Here, $M = 2x\sinh\left(\frac{y}{x}\right) + 3y\cosh\left(\frac{y}{x}\right)$ and $N = -3x\cosh\left(\frac{y}{x}\right)$

$\therefore \frac{\partial M}{\partial y} = 2x \cdot \cosh\left(\frac{y}{x}\right)\cdot\frac{1}{x} + 3\cosh\left(\frac{y}{x}\right) + 3y\sinh\left(\frac{y}{x}\right)\cdot\frac{1}{x} \quad = 5\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sinh\left(\frac{y}{x}\right)$

$\therefore \frac{\partial N}{\partial x} = -3\cosh\left(\frac{y}{x}\right) - 3x\sinh\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) \quad = -3\cosh\left(\frac{y}{x}\right) + \frac{3y}{x}\sin\left(\frac{y}{x}\right)$

The DE is not exact.

$\therefore \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N} = \frac{8\cosh\left(\frac{y}{x}\right)}{-3x\cosh\left(\frac{y}{x}\right)} = -\frac{8}{3x} = f(x)$

$\therefore IF = e^{\int f(x)dx} = e^{\int -(8/3x)dx} = e^{-(8/3)\log x} = x^{-(8/3)}$

Multiply the given equation by IF, we get,

$\left[2x^{-5/3}\cdot\sinh\left(\frac{y}{x}\right) + 3x^{-8/3}\cdot y\cdot\cosh\left(\frac{y}{x}\right)\right]dx - 3x^{-5/3}\cdot\cosh\left(\frac{y}{x}\right)\cdot dy = 0$

Since $\int M\,dx$ is rather a complex integral

$\therefore \int N\,dy$ treating x constant

$= \int -3x^{-5/3}\cdot\cosh\left(\frac{y}{x}\right)dy = -3x^{-5/3}\cdot\sinh\left(\frac{y}{x}\right)\cdot x = -3x^{-2/3}\cdot\sinh\left(\frac{y}{x}\right)$

$\int(\text{terms in } M \text{ free from } y)\,dx = 0$

$\therefore$ The solution is $x^{-2/3}\cdot\sinh\left(\frac{y}{x}\right) = -\frac{c\prime}{3} = c$

5. Solve $(x^2 - xy + y^2)dx - xydy = 0$

Solution: The DE is not exact. (show this !!!)

The given differential equation is homogeneous

and $Mx + Ny = x^3 - x^2y + xy^2 - xy^2 = x^2(x-y)$

$\therefore \frac{1}{(Mx+Ny)} = \frac{1}{x^2(x-y)}$ is an integrating factor

Multiply the given equation by IF, we get, $\frac{x^2-xy+y^2}{x^2(x-y)}dx - \frac{xy}{x^2(x-y)}dy = 0$

$\therefore \frac{x^2-xy}{x^2(x-y)}dx + \frac{y^2}{x^2(x-y)}dx - \frac{y}{x(x-y)}dy = 0$

$\therefore \left[\frac{1}{x} + \frac{1}{x-y} - \frac{1}{x} - \frac{y}{x^2}\right]dx + \left[\frac{1}{x} - \frac{1}{x-y}\right]dy = 0$ (By partial fraction)

$\therefore \left[\frac{1}{x-y} - \frac{y}{x^2}\right]dx + \left[\frac{1}{x} - \frac{1}{x-y}\right]dy = 0$

$\therefore \int M\,dx = \int\frac{dx}{x-y} - \int\frac{y}{x^2}dx = \log(x-y) + \frac{y}{x}$

$\int$(terms in N free from x) $dy = 0$

$\therefore$ The solution is $\log(x-y) + \frac{y}{x} = c$

6. $(xy\,sin\,xy\ +\ cos\,xy)\,y\,dx\ +\ (xy\,sin\,xy - cos\,xy)\,x\,dy\ =\ 0$

Solution: We have, $M = (xy\,sin\,xy\ +\ cos\,xy)\,y$ and $N =\ (xy\,sin\,xy - cos\,xy)\,x$

The DE is not exact. (show this !!!)

$Mx - Ny = x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy = 2xy\cos xy$

$\therefore IF = \frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$

Multiply the given equation by IF, we get, $\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0$

$\therefore \left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$, which is exact

$\therefore \int M\,dx = \int\left(y\tan xy + \frac{1}{x}\right)dx = \log\sec xy + \log x$

$\int$(terms in N free from x) $dy = \int -\frac{1}{y}dy = -\log y$

$\therefore$ The solution is $\log\sec xy + \log x = \log y + \log c$ i.e. $x\sec xy = cy$

7. $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy\ =\ 0$

Solution : The DE is not exact. (show this !!!)

The given differential equation is homogeneous

and $Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2$

Hence, $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ is an integrating factor

Multiply the given equation by IF, we get, $\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ which is exact

Now, $\int M\,dx = \int\left(\frac{1}{y} - \frac{2}{x}\right)dx = \frac{x}{y} - 2\log x$

And $\int$(terms in N free from x) $dy = \int\frac{3}{y}dy = 3\log y$

$\therefore$ The solution is $\frac{x}{y} - 2\log x + 3\log y = -\log c$

i.e. $\frac{x}{y} + \log\frac{cy^3}{x^2} = 0$ i.e. $\log\frac{cy^3}{x^2} = -\frac{x}{y}$ i.e. $\frac{cy^3}{x^2} = e^{-x/y}$

8. If $f(x)$ a function of x only is an integrating factor of $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0$. find $f(x)$ and then solve the equation.

Solution: The given equation is $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0$

Multiplying the equation by $f(x)$,

$f(x)\cdot(x^4e^x - 2mxy^2)dx + f(x)\cdot 2mx^2ydy = 0$

$\therefore M = f(x)\cdot(x^4e^x - 2mxy^2),\quad N = f(x)\cdot 2mx^2y$

$\therefore \frac{\partial M}{\partial y} = f(x)(-4m\,xy),\qquad \frac{\partial N}{\partial x} = f(x)4m\,xy + f'(x)2mx^2y$

Since, now equation is exact $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore f(x)(-4m\,xy) = f(x)4m\,xy + f'(x)2mx^2y$

$\therefore -4mxy\cdot f(x) = f'(x)\cdot mx^2y$

$\therefore f'(x) = -\frac{4}{x}\,f(x) \qquad \therefore \frac{f'(x)}{f(x)} = -\frac{4}{x} \qquad \therefore \int\frac{f'(x)}{f(x)}dx = -4\int\frac{dx}{x}$

$\therefore \log f(x) = -4\log x = \log x^{-4}$

$\therefore f(x) = x^{-4}$

Now, multiply the given equation by x^{-4} so that it becomes exact and solve as above.

9. If $(x+y)^k$ is an integrating factor of $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$. find k and solve the equation.

Solution: The given equation is $(4x^2 + 2xy + 6y)\,dx + (2x^2 + 9y + 3x)\,dy = 0$.

Multiplying the equation by $(x+y)^k$, we get,

$(x+y)^k(4x^2 + 2xy + 6y)dx + (x+y)^k(2x^2 + 9y + 3x)dy = 0$ …………… (1)

$\therefore M = (x+y)^k(4x^2 + 2xy + 6y), \quad N = (x+y)^k(2x^2 + 9y + 3x)$

$\therefore \frac{\partial M}{\partial y} = k(x+y)^{k-1}(4x^2 + 2xy + 6y) + (x+y)^k(2x+6),$

$\frac{\partial N}{\partial x} = k(x+y)^{k-1}(2x^2 + 9y + 3x) + (x+y)^k(4x+3)$

Since, now equation is exact $\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore k(x+y)^{k-1}(4x^2 + 2xy + 6y) + (x+y)^k(2x+6)$

$= k(x+y)^{k-1}(2x^2 + 9y + 3x) + (x+y)^k(4x+3)$

$\therefore k(4x^2 + 2xy + 6y) + (x+y)(2x+6) = k(2x^2 + 9y + 3x) + (x+y)(4x+3)$

$\therefore 4kx^2 + 2kxy + 6ky + 2x^2 + 6x + 2xy + 6y = 2kx^2 + 9ky + 3kx + 4x^2 + 3x + 4xy + 3y$

$\therefore k(2x^2 + 2xy - 3y - 3x) = (2x^2 + 2xy - 3y - 3x)$ $\therefore k = 1$

Putting $k = 1$ in (1), we get,

$(x+y)(4x^2 + 2xy + 6y)dx + (x+y)(2x^2 + 9y + 3x)dy = 0$ which is exact

$(4x^3 + 2x^2y + 6xy + 4x^2y + 2xy^2 + 6y^2)dx + (2x^3 + 9xy + 3x^2 + 2x^2y + 9y^2 + 3xy)dy = 0$

$\therefore (4x^3 + 6x^2y + 6xy + 2xy^2 + 6y^2)dx + (2x^3 + 12xy + 3x^2 + 2x^2y + 9y^2)dy = 0$

$\therefore \int M\,dx = \int (4x^3 + 6x^2y + 6xy + 2xy^2 + 6y^2)\,dx = x^4 + 2x^3y + 3x^2y + x^2y^2 + 6y^2x$

$\int(\text{Terms in } M \text{ free from } y)\,dy = \int 9y^2\,dy = 3y^3$

$\therefore$ The solution is $x^4 + 2x^3y + 3x^2y + x^2y^2 + 6y^2x + 3y^3 = c$