Cauchy's Equation

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- **Definition:** An equation of the form
- $x^n \frac{d^n y}{d x^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_{n-1} x \frac{dy}{dx} + p_n y = X$
- where $p_1, p_2 \ldots\ldots, p_n$ are constants and X is a function of x is called **homogeneous linear differential equation of order n**.
- The equation is also known as **Cauchy's equation.**

for example $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 5y = \sin(\log x)$

↓ substitution

$\rightarrow \frac{d^2y}{dx^2} - P_1 \frac{dy}{dx} + P_2 y = X$

METHOD OF SOLUTION

- The equation can be transformed into an equation with constant coefficients by the substitution $z = \log x$ **or** $x = e^z$
- Now, $\because z = \log x, \frac{dz}{dx} = \frac{1}{x}$ *and*
- $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$ ....(i)
- $\frac{d^2y}{dx^2} = -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} = -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x^2} \cdot \frac{d^2y}{dz^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$ ....(ii)

$y \rightarrow z \rightarrow x$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$

$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz} \rightarrow x \frac{dy}{dx} = \frac{dy}{dz}$

$\frac{d^2y}{dx^2} = \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$

$\frac{d^2y}{dx^2} = \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

- Similarly, it can be shown that $\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)$ ....(iii) and so on.
- If we put $D = \frac{d}{dz}$ then we get, from (i), (ii), (iii),

$\rightarrow x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$

- Similarly, it can be shown that $\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right)$ ....(iii) and so on.
- If we put $D = \frac{d}{dz}$ then we get, from (i), (ii), (iii), $\rightarrow$ $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}$

  $= D^3y - 3D^2y + 2Dy$
- $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$
- $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)y$
- $x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} = D^3y - 3D^2y + 2Dy = D(D-1)(D-2)y$ and so on.

$x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

$x^4\frac{d^4y}{dx^4} = D(D-1)(D-2)(D-3)y$

- Further the r.h.s. X by the substitution of $x = e^z$ changes to a function of z only say Z.
- Thus, the given equation by the substitution $x = e^z$ changes to a linear differential equation with constant coefficients of the form $f(D)y = Z$ and can be solved by the methods studied in the previous exercise.

EXAMPLE-1: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x)$

Soln :- Put $z = \log x$ ie $x = e^z$ and $\frac{d}{dz} = D$ we get

$x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

The given D.E. changes to

$D(D-1)y - 3Dy + 5y = \sin z$

$D^2y - 4Dy + 5y = \sin z$

$(D^2 - 4D + 5)y = \sin z$

A.E is $m^2 - 4m + 5 = 0$

$m = 2 \pm i$

$\therefore y_c = e^{2z}(c_1\cos z + c_2\sin z)$

$P.I = y_p = \frac{1}{D^2 - 4D + 5}\sin z$

Sub $D^2 = -(1)^2 = -1$

$$= \frac{1}{4-4D} \sin z$$

$$= \frac{1(1+D)}{4(1-D)(1+D)} \sin z$$

$$= \frac{1}{4}(1+D) \cdot \frac{1}{1-D^2} \sin z$$

Put $D^2 = -1$

$$= \frac{1}{4}(1+D) \cdot \frac{1}{1-(-1)} \sin z = \frac{1}{8}(1+D) \sin z$$

$$y_p = \frac{1}{8}(\sin z + \cos z)$$

$$\therefore y = y_c + y_p = e^{2z}(c_1 \cos z + c_2 \sin z) + \frac{1}{8}(\sin z + \cos z)$$

resubstituting $z = \log x$

$$y = x^2 \left[c_1 \cos(\log x) + c_2 \sin(\log x) \right] + \frac{1}{8}\left[\sin(\log x) + \cos(\log x) \right]$$

EXAMPLE-2: $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3 + 3x.$

Soln :- put $z = \log x$ ie $x = e^z$ and $D = \frac{d}{dz}$ we get

$$x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y, \quad x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

The given D.E. changes to

$$D(D-1)(D-2)y - D(D-1)y + 2Dy - 2y = e^{3z} + 3e^{z}$$

$$(D^3 - 4D^2 + 5D - 2)y = e^{3z} + 3e^{z}$$

A.E is $\quad m^3 - 4m^2 + 5m - 2 = 0$

$$m = 1, 1, 2$$

The C.F is $\quad y_c = (c_1 + c_2 z)e^{z} + c_3 e^{2z}$

$$P.I. = y_p = \frac{1}{D^3 - 4D^2 + 5D - 2}\left(e^{3z} + 3e^{z}\right)$$

$$= \frac{1}{D^3 - 4D^2 + 5D - 2} e^{3z} + 3 \cdot \frac{1}{D^3 - 4D^2 + 5D - 2} e^{z}$$

put $D = 3$

put $D = 1$
denominater becomes zero

$$= \frac{1}{4} e^{3z} + \frac{3 \cdot z}{3D^2 - 8D + 5} e^{z}$$

put $D = 1$, denominater $= 0$

$$= \frac{1}{4} e^{3z} + \frac{3 \cdot z^2}{6D - 8} e^{z}$$

put $D = 1$

$$y_p = \frac{1}{4} e^{3z} - \frac{3}{2} z^2 e^{z}$$

$$\therefore \quad y = y_c + y_p$$

$$= (c_1 + c_2 z)e^{z} + c_3 e^{2z} + \frac{1}{4} e^{3z} - \frac{3}{2} z^2 e^{z}$$

$$= (c_1 + c_2 z)e^{z} + c_3 e^{2z} + \frac{1}{4}e^{3z} - \frac{3}{2}z^2 e^{z}$$

Resubstituting $z = \log x$, $x = e^{z}$

$$y = [c_1 + c_2(\log x)]x + c_3 x^2 + \frac{1}{4}x^3 - \frac{3}{2}x(\log x)^2$$

EXAMPLE-3: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{(\sin\log x)+1}{x}$

Soln:- put $z = \log x$ ie $x = e^{z}$, $D = \frac{d}{dz}$ we got

$$x\frac{dy}{dx} = Dy, \quad x^2\frac{d^2y}{dx^2} = D(D-1)y$$

The given D.E. changes to

$$D(D-1)y - 3Dy + y = \frac{\sin z + 1}{e^{z}} = e^{-z}(\sin z + 1)$$

$$(D^2 - 4D + 1)y = e^{-z}\sin z + e^{-z}$$

A.E is $m^2 - 4m + 1 = 0$

$m = 2 \pm \sqrt{3}$

$$y_c = e^{2z}[c_1\cosh(\sqrt{3}z) + c_2\sinh(\sqrt{3}z)]$$

$$y_p = \frac{1}{D^2-4D+1}e^{-z}\sin z + \frac{1}{D^2-4D+1}e^{-z}$$

$$\frac{1}{D^2-4D+1}e^{-z} = \frac{1}{6}e^{-z}$$

put $z = -1$

$$\frac{1}{\quad}e^{-z}\sin z = e^{-z}\cdot\frac{1}{\quad}\sin z$$

$$\frac{1}{D^2-4D+1}e^{-z}\sin z = e^{-z}\cdot\frac{1}{(D-1)^2-4(D-1)+1}\sin z$$

$$= e^{-z}\cdot\frac{1}{D^2-6D+6}\sin z$$

put $D^2=-1$

$$= e^{-z}\cdot\frac{1}{5-6D}\sin z$$

$$= e^{-z}\,\frac{(5+6D)}{25-36D^2}\sin z$$

put $D^2=-1$

$$= \frac{e^{-z}}{61}(5+6D)\sin z$$

$$= \frac{e^{-z}}{61}\left[5\sin z+6\cos z\right]$$

$$\therefore y_p = \frac{e^{-z}}{61}\left[5\sin z+6\cos z\right]+\frac{1}{6}e^{-z}$$

$$y = y_c + y_p$$

$$= e^{2z}\left[c_1\cosh(\sqrt{3}z)+c_2\sinh(\sqrt{3}z)\right]$$

$$-\frac{e^{-z}}{61}\left[5\sin z+6\cos z\right]+\frac{1}{6}e^{-z}$$

$$z=\log x,\quad x=e^z$$

$$y = x^2\left[C_1 \cosh\left(\sqrt{3}\log x\right) + C_2 \sinh\left(\sqrt{3}\log x\right)\right]$$
$$- \frac{1}{61x}\left[5\sin(\log x) + 6\cos(\log x)\right] + \frac{1}{6x}$$

EXAMPLE-4: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x.\sin(\log x)$