METHOD OF VARIATION OF PARAMETERS

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This is one of the methods for finding the Particular Integral (P.I.) of a linear differential equation whose Complimentary function (C.F.) is known.
Though the method is general, we will illustrate it by applying it to a second order and third order differential equation.

(1) Consider the linear equation of second order with constant coefficients. $aD^2y + bDy + cy = X$

i.e. $(aD^2 + bD + c)\,y = X$ $\quad f(D)y = X$

$m = 1, 2$

$C.F = c_1 e^x + c_2 e^{2x}$

$y_1 = e^x$

$y_2 = e^{2x}$

Let Complementary function $=$ $c_1\,y_1 + c_2 y_2$ then Particular Integral $=$ $uy_1 + vy_2$ where

$u = -\int \frac{y_2 X}{W} dx$ & $V = \int \frac{y_1 X}{W} dx$ & $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow$ wronskin

∴ General solution = Complementary function + Particular Integral.

① $\frac{d^2y}{dx^2} + a^2 y = \sec ax$

A.E is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore$ C.F $= y_c = c_1 \cos ax + c_2 \sin ax$

$= c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos ax, \quad y_2 = \sin ax$

let $PI = uy_1 + vy_2$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} =$

$= a(\cos^2 ax + \sin^2 ax)$

$W = a$

Now $u = -\int \frac{y_2 X}{W} dx = -\int \frac{\sin ax \cdot \sec ax}{a} dx$

$= \frac{-1}{a}\int \tan ax\, dx = \frac{-1}{a}\left[\frac{\log|\sec ax|}{a}\right]$

$= \frac{1}{a^2} \log|\cos ax|$

$$V = \int \frac{y_1 X}{W} dx = \int \frac{\cos ax \cdot \sec ax}{a} dx$$

$$= \frac{1}{a}\int dx = \frac{x}{a}$$

$$\therefore y_p = PI = u y_1 + v y_2 = \frac{1}{a^2}\log|\cos ax| \cdot \cos ax$$
$$+ \frac{x}{a}\sin ax$$

$\therefore$ Complete solution $= y_c + y_p$

$$= c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}\cos ax \cdot \log|\cos ax|$$
$$+ \frac{x}{a}\sin ax$$

(2) $(D^2 - 2D + 2)y = e^x \tan x$

Solⁿ: A.E. $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

$$\therefore CF = y_c = e^x(c_1 \cos x + c_2 \sin x)$$
$$= c_1 e^x \cos x + c_2 e^x \sin x$$
$$= c_1 y_1 + c_2 y_2$$

$y_1 = e^x \cos x \qquad y_2 = e^x \sin x$

Let $y_p = PI = u y_1 + v y_2$

Now $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$

$$= e^{2x}(\sin x \cos x + \cos^2 x) - e^{2x}(\sin x \cos x - \sin^2 x)$$
$$= e^{2x}(\cos^2 x + \sin^2 x) = e^{2x}$$

Now $u = -\int \frac{y_2 X}{W} dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}} dx$

$$= -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int \sec x\,dx + \int \cos x\,dx$$

$$u = -\log|\sec x + \tan x| + \sin x$$

Also, $V = \int \frac{y_1 X}{W} = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx$

$$V = \int \sin x\,dx = -\cos x$$

$\therefore\ y_p = u y_1 + V y_2$

$$= e^x \cos x \left[-\log|\sec x + \tan x| + \sin x\right]$$

$$+ e^x \sin x\,(-\cos x)$$

$$y_p = -e^x \cos x \log|\sec x + \tan x|$$

$\therefore$ complete sol^n = $y_c + y_p$

$$= c_1 e^x \cos x + c_2 e^x \sin x - e^x \cos x \log|\sec x + \tan x|$$

(3) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$

A.E. $\rightarrow$ $m^2 + 3m + 2 = 0$

$m = -1, -2$

$CF = y_c = c_1 e^{-x} + c_2 e^{-2x}$

$y_1 = e^{-x}$ $\quad y_2 = e^{-2x}$

$\therefore y_1 = e^{-x}, \quad y_2 = e^{-2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x}$$
$$= -e^{-3x}$$

$$u = -\int \frac{y_2 X}{W}\,dx = -\int \frac{e^{-2x}\, e^{e^x}}{-e^{-3x}}\,dx = \int e^x . e^{e^x}\,dx$$

put $e^x = t \Rightarrow e^x\,dx = dt$

$$u = \int e^t\,dt = e^t = e^{e^x}$$

$$v = \int \frac{y_1 X}{W}\,dx = \int \frac{e^{-x} . e^{e^x}}{-e^{-3x}}\,dx = -\int e^{2x} . e^{e^x}\,dx$$

put $e^x = t \Rightarrow e^x\,dx = dt$

$$v = -\int t e^t\,dt = -\left[t(e^t) - \int (1)\, e^t\,dt\right]$$
$$= -\left[te^t - e^t\right] = -\left[e^x e^{e^x} - e^{e^x}\right]$$

$$v = -e^{e^x}(e^x - 1)$$

$$\therefore y_p = u y_1 + v y_2 = e^{e^x} . e^{-x} - e^{e^x}(e^x - 1)\, e^{-2x}$$
$$= e^{e^x}\left[e^{-x} - e^{-x} + e^{-2x}\right]$$

$$y_p = e^{-2x} . e^{e^x}$$

$$\therefore y = y_c + y_2$$
$$= c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} . e^{e^x}$$

(2) Consider the linear equation of third order with constants coefficient $aD^3y + bD^2y + cDy + dy = X$

i.e. $(aD^3 + bD^2 + cD + d)\,y = X$ $\quad f(D)y = X$

Let Complementary function $= c_1\,y_1 + c_2y_2 + c_3\,y_3$ then Particular Integral $= uy_1 + vy_2 + wy_3$ where

$u = \int \frac{(y_2y_3' - y_3y_2')X}{W}dx, \qquad v = \int \frac{(y_3y_1' - y_1y_3')X}{W}dx, \qquad w = \int \frac{(y_1y_2' - y_2y_1')X}{W}dx$

Where $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$

$\therefore$ General Solution = Complementary function + Particular Integral.

① $(D^3 + 4D)\,y = 4\cot 2x$

Soln: A.E. $m^3 + 4m = 0 \Rightarrow m = 0, \pm 2i$

$\therefore y_c = CF = c_1 e^{0x} + c_2 \cos 2x + c_3 \sin 2x$

$= c_1 + c_2 \cos 2x + c_3 \sin 2x$

$= c_1 y_1 + c_2 y_2 + c_3 y_3$

$\therefore y_1 = 1, \quad y_2 = \cos 2x, \quad y_3 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8(\cos^2 2x + \sin^2 2x) = 8$$

$PI = y_p = uy_1 + vy_2 + wy_3$

Now $u = \int \frac{(y_2 y_3' - y_3 y_2')X}{W}dx$

$$= \int \frac{[\cos 2x \cdot (2\cos 2x) - \sin 2x(-2\sin 2x)]\,4\cot 2x}{8}dx$$

$$u = \int \cot 2x\,dx = \frac{1}{2}\log|\sin 2x|$$

$$v = \int \frac{(y_3 y_1' - y_1 y_3')}{W} \, X \, dx =$$

$$= \int \frac{[\sin 2x (0) - (1)(2\cos 2x)]}{8} \cdot 4\cot 2x \, dx$$

$$= -\int \frac{\cos^2 2x}{\sin 2x} \, dx = -\int \frac{1 - \sin^2 2x}{\sin 2x} \, dx$$

$$= -\int \operatorname{cosec} 2x \, dx + \int \sin 2x \, dx$$

$$v = -\frac{1}{2} \log |\operatorname{cosec} 2x - \cot 2x| - \frac{1}{2} \cos 2x$$

$$w = \int \frac{(y_1 y_2' - y_2 y_1')}{W} \cdot X \, dx$$

$$= \int \frac{[(1)(-2\sin 2x) - \cos 2x (0)]}{8} \cdot 4\cot 2x \, dx$$

$$w = -\int \cos 2x \, dx = -\frac{1}{2} \sin 2x$$

$$\therefore y_p = u y_1 + v y_2 + w y_3$$

$$= \frac{1}{2} \log |\sin 2x| + \cos 2x \cdot \left(-\frac{1}{2}\right) \left[\log |\operatorname{cosec} 2x - \cot 2x| + \cos 2x\right]$$

$$-\frac{1}{2} \sin^2 2x$$

$$= \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}(\cos^2 2x + \sin^2 2x)$$

$$y_p = \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}$$

$\therefore$ complete solution

$$y = y_c + y_p$$
$$= c_1 + c_2 \cos 2x + c_3 \sin 2x$$
$$+ \frac{1}{2}\log|\sin 2x| - \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$- \frac{1}{2}$$

$$y = C + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2}\log|\sin 2x|$$
$$- \frac{1}{2}\cos 2x \log|\operatorname{cosec} 2x - \cot 2x|$$
$$\left[C = c_1 - \frac{1}{2}\right]$$