General method to find PI

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e^{ax}, $\sin ax$, $\cos ax$, x^m, $e^{ax}\cdot V$

- **Case (v) When the r.h.s. X does not belong partially or completely to any one of the above forms.**

When any of the above method fails to give the Particular Integral we apply the definition of $\frac{1}{f(D)}X$ i.e.

$\frac{1}{D}X = \int X\,dx$,

$\frac{1}{D-a}X = e^{ax}\int e^{-ax}X\,dx$,

$\frac{1}{D+a}X = e^{-ax}\int e^{ax}X\,dx$

$$\frac{1}{D-3}\left(\frac{1}{x}\right) \rightarrow e^{3x}\int e^{-3x}\cdot\frac{1}{x}\,dx$$

$$\frac{1}{D+3}(\log x) \rightarrow e^{-3x}\int e^{3x}(\log x)\,dx$$

$$\frac{1}{D^2-3D+2}(\log x) = \frac{1}{(D-1)(D-2)}(\log x)$$

either $= \frac{1}{D-1}\left[e^{2x}\int e^{-2x}\log x\,dx\right]$

or $\left[-\frac{1}{D-1}+\frac{1}{D-2}\right](\log x)$

$$-e^{x}\int e^{-x}\log x\,dx + e^{2x}\int e^{-2x}\log x\,dx$$

Example – 1: $(D^2+a^2)y = \sec ax$

Soln :- A.E is $m^2+a^2=0$

$m = \pm ai$

C.F is $y_c = c_1\cos ax + c_2\sin ax$

$$y_p = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{}\sec ax$$

$$= \frac{\qquad}{(D+ai)(D-ai)} \ldots$$

$$= \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax$$

$$\frac{1}{D-a}X = e^{ax}\int e^{-ax} X\,dx \quad \text{and} \quad \frac{1}{D+a}X = e^{-ax}\int e^{ax} X\,dx$$

$$\therefore y_p = \frac{1}{2ai}\left[e^{aix}\int e^{-aix} \sec ax\,dx - e^{-aix}\int e^{aix} \sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int (\cos ax - i\sin ax)\sec ax\,dx \right.$$

$$\left. - e^{-aix}\int (\cos ax + i\sin ax)\sec ax\,dx\right]$$

$$= \frac{1}{2ai}\left[e^{aix}\int (1 - i\tan ax)\,dx - e^{-aix}\int (1 + i\tan ax)\,dx\right]$$

$$= \frac{1}{2ai}\left\{e^{aix}\left[x - \frac{i}{a}\log \sec ax\right] - e^{-aix}\left[x + \frac{i}{a}\log \sec ax\right]\right\}$$

$$= \frac{1}{2ai}\left\{(\cos ax + i\sin ax)\left(x - \frac{i}{a}\log \sec ax\right)\right.$$

$$\left. - (\cos ax - i\sin ax)\left(x + \frac{i}{a}\log \sec ax\right)\right\}$$

$$y_p = \frac{1}{2ai}\left\{2ix\sin ax - \frac{2i}{a}\cos ax \log \sec ax\right\}$$

$$y_p = \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax \log \sec ax$$

$$\therefore y = y_c + y_p$$

$$= c_1\cos ax + c_2 \sin ax + \frac{x}{a}\sin ax - \frac{1}{a^2}\cos ax(\log \sec ax)$$

EXAMPLE-2:

- $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$

<u>Solution</u> A.E is $m^2+5m+6=0$

$$m = -2, -3$$

$$\therefore \text{C.F is } y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y_p = \frac{1}{(D+2)(D+3)} e^{-2x}\sec^2 x\,(1+2\tan x)$$

$$\frac{1}{D+3}\cdot\frac{1}{D+2}(X) = \frac{1}{D+3}\cdot e^{-2x}\int e^{2x}\cdot X\,dx$$

$$= \frac{1}{D+3}\cdot e^{-2x}\int e^{2x}\left(e^{-2x}\sec^2 x\,(1+2\tan x)\right)dx$$

$$= \frac{1}{D+3}\cdot e^{-2x}\int(\sec^2 x + 2\tan x\sec^2 x)\,dx$$

$$= \frac{1}{D+3}\cdot e^{-2x}\left[\tan x + \tan^2 x\right]$$

$$= e^{-3x}\int e^{3x}\cdot e^{-2x}(\tan x + \tan^2 x)\,dx$$

$$= e^{-3x}\int e^{x}\left[\tan x + \tan^2 x\right]dx$$

$$= e^{-3x}\int e^{x}\left[\tan x + \sec^2 x - 1\right]dx$$

$$= e^{-3x}\int e^{x}\left[\tan x + \sec^2 x - 1\right] dx$$

$$= e^{-3x}\left\{\int e^{x}(\tan x + \sec^2 x)\,dx - \int e^{x}\,dx\right\}$$

$$\int e^{x}\left[f(x)+f'(x)\right]dx = e^{x}f(x)$$

$$= e^{-3x}\left\{e^{x}\tan x - e^{x}\right\}$$

$$y_p = e^{-2x}(\tan x - 1)$$

Complete solutions is

$$y = y_c + y_p = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1)$$

EXAMPLE-3:

- $(D^2 + D)y = \frac{1}{1+e^x}$

Soln :-

$$m^2 + m = 0$$

$$m = 0, -1$$

$$y_c = c_1 + c_2 e^{-x}$$

$$y_p = \frac{1}{(D+1)D}\,\frac{1}{1+e^x}$$

$$= \frac{1}{D+1}\cdot\frac{1}{D}\,\frac{1}{1+e^x}$$

$$= \frac{1}{D+1}\int \frac{1}{1+e^x}\,dx$$

$$= \frac{1}{D+1}\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$= \frac{1}{D+1}\int \frac{1}{1+e^{-x}}\,dx$$

put $e^{-x}+1 = t$

$-e^{-x}\,dx = dt$

$$y_p = \frac{1}{D+1}\int -\frac{1}{t}\,dt$$

$$\frac{1}{D+a}X = e^{-ax}\int e^{ax}X\,dx$$

$$= \frac{1}{D+1}\left[-\log(e^{-x}+1)\right]$$

$$= -e^{-x}\int e^{x}\log(e^{-x}+1)\,dx$$

Integrating by parts

$$= -e^{-x}\left[\log(e^{-x}+1)(e^{x}) - \int \frac{-e^{-x}}{e^{-x}+1}e^{x}\,dx\right]$$

$$= -e^{-x}\left[e^{x}\log(e^{-x}+1) + \int \frac{1}{e^{-x}+1}\,dx\right]$$

$$= -e^{-x}\left[e^{x}\log(e^{-x}+1) + \int \frac{e^{x}}{e^{x}+1}\,dx\right]$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log[f(x)]$$

$$y_p = -e^{-x}\left[e^{x}\log(e^{-x}+1) + \log(e^{x}+1)\right]$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$= c_1 + c_2e^{-x} - e^{-x}\left[e^{x}\log(e^{-x}+1) + \log(e^{x}+1)\right]$$

EXAMPLE-4: $(D^2 - D - 2)y = 2\log x + \frac{1}{x} + \frac{1}{x^2}$

Sol^n: $(D^2 - D - 2) = 0$

$$m^2 - m - 2 = 0$$

$$m = -1, 2$$

$$y_c = c_1 e^{-x} + c_2 e^{2x}$$

$$y_p = \frac{1}{(D-2)(D+1)}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right)$$

$$= \frac{1}{D-2}\, e^{-x} \int e^{x}\left(2\log x + \frac{1}{x} + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{D-2}\, e^{-x}\left[\int e^{x}\left(2\log x + \frac{2}{x}\right) dx + \int e^{x}\left(-\frac{1}{x} + \frac{1}{x^2}\right) dx\right]$$

$$\int e^{x}\left[f(x) + f'(x)\right] dx = e^{x} f(x)$$

$$= \frac{1}{D-2}\, e^{-x}\left[e^{x}(2\log x) + e^{x}\left(-\frac{1}{x}\right)\right]$$

$$= \frac{1}{D-2}\left(2\log x - \frac{1}{x}\right)$$

$$= e^{2x} \int e^{-2x}\left(2\log x - \frac{1}{x}\right) dx$$

$$= e^{2x}\int e^{-2x}\left(2\log x - \frac{1}{x}\right)dx$$

$$\int e^{ax}\left[af(x) + f'(x)\right]dx = e^{ax}f(x)$$

$$\int e^{-ax}\left[af(x) - f'(x)\right]dx = -e^{-ax}f(x)$$

$$= e^{2x}\left[-e^{-2x}\log x\right]$$

$$y_p = -\log x$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{2x} - \log x$$