METHOD TO FIND PI WHEN RHS= $e^{ax}\,V$

- **Case (iv) When the r.h.s. $X = e^{ax}\,V$ where V is a function of x.**
- $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$

EXAMPLE − 1: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = (x^2e^x)^2$

Soln :-

$$(D^2 - 4D + 3)y = x^4 e^{2x}$$

A.E is $m^2 - 4m + 3 = 0$

$m = 1, 3$

C.F is $y_c = c_1 e^{x} + c_2 e^{3x}$

$$y_p = \frac{1}{D^2 - 4D + 3} e^{2x}.x^4$$

$$= e^{2x}\frac{1}{(D+2)^2 - 4(D+2) + 3}x^4$$

$$= e^{2x}.\frac{1}{D^2 + 4D + 4 - 4D - 8 + 3}x^4$$

$$= e^{2x}.\frac{1}{D^2 - 1}x^4$$

$$= -e^{2x}.\frac{1}{1 - D^2}x^4$$

$$\frac{1}{f(D)}e^{ax}.V$$

$$= e^{ax}.\frac{1}{f(D+a)}V$$

$$= -e^{2x}\left(1-D^2\right)^{-1} x^4$$

$$\left[\text{wkt } (1-t)^{-1} = 1+t+t^2+t^3+\cdots\cdots\right]$$

$$y_p = -e^{2x}\left[1+D^2+D^4+D^6+\cdots\right]x^4$$

$$= -e^{2x}\left(1+D^2+D^4\right)x^4$$

$$y_p = -e^{2x}\left(x^4+12x^2+24\right)$$

$\therefore$ The complete soln is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{3x} - e^{2x}\left(x^4+12x^2+24\right)$$

EXAMPLE – 2: $(D^3+1)y = e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$

Soln :- A·E is $m^3+1=0$

$$m = -1,\ \frac{1 \pm i\sqrt{3}}{2}$$

$$y_c = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p = \frac{1}{D^3+1}e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{x/2}\cdot\frac{1}{\left(D+\frac{1}{2}\right)^3+1}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{\quad}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$D^3+\frac{3}{2}D^2+\frac{3}{4}D+\frac{9}{8}$$

put $D^2=-\left(\frac{\sqrt{3}}{2}\right)^2=\frac{-3}{4}$

denominator becomes zero

$$=e^{\frac{x}{2}}\cdot\frac{x}{3D^2+3D+\frac{3}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

put $D^2=-\left(\frac{\sqrt{3}}{2}\right)^2=\frac{-3}{4}$

$$\therefore\ y_p=e^{\frac{x}{2}}\ \frac{x}{3D-\frac{3}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$=e^{\frac{x}{2}}\cdot\frac{x\left(3D+\frac{3}{2}\right)}{9D^2-\frac{9}{4}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$=\frac{e^{\frac{x}{2}}\cdot x\left[3\left(\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\sqrt{3}}{2}x\right)+\frac{3}{2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right]}{9\left(\frac{-3}{4}\right)-\frac{9}{4}}$$

$$\therefore\ y_p=-\frac{xe^{\frac{x}{2}}}{6}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$\therefore$ complete solution is

$$y=y_c+y_p$$

$$=c_1e^{-x}+e^{\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$-\frac{xe^{\frac{x}{2}}}{6}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)+\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

- $EXAMPLE-3\text{:}\ (D^2+2)y=e^x\cos x+x^2e^{3x}$

Sol^n:- A·E is $m^2+2=0$

$m = \pm\sqrt{2}\,i$

$\therefore$ CF is $y_c = c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)$

$$y_p = \frac{1}{D^2+2}\left(e^x\cos x + x^2 e^{3x}\right)$$

$$= \frac{1}{D^2+2}e^x\cos x + \frac{1}{D^2+2}e^{3x}x^2$$

$$\left[\frac{1}{f(D)}e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}.V\right]$$

$$y_p = e^x.\frac{1}{(D+1)^2+2}\cos x + e^{3x}.\frac{1}{(D+3)^2+2}x^2$$

$$= e^x.\frac{1}{D^2+2D+3}\cos x + e^{3x}.\frac{1}{D^2+6D+11}x^2 \quad \text{--- (1)}$$

(put $D^2=-1$)

$$\frac{1}{D^2+2D+3}\cos x = \frac{1}{-1+2D+3}\cos x = \frac{1}{2D+2}\cos x$$

$$= \frac{1\,(D-1)}{2(D+1)(D-1)}\cos x = \frac{D-1}{2(D^2-1)}\cos x$$

(put $D^2=-1$)

$$= \frac{-1}{4}(D-1)\cos x$$

$$= \frac{-1}{4}(-\sin x - \cos x) = \frac{1}{4}(\sin x + \cos x)$$

Also $\dfrac{1}{D^2+6D+11}x^2 = \dfrac{1}{11\left[1+\dfrac{D^2+6D}{11}\right]}x^2$

$$= \frac{1}{11}\left[1+\frac{D^2+6D}{11}\right]^{-1}x^2$$

$$\left[(1+t)^{-1} = 1-t+t^2-t^3\cdots\cdots\right]$$

$$= \frac{1}{11}\left[1-\left(\frac{D^2+6D}{11}\right)+\left(\frac{D^2+6D}{11}\right)^2+\cdots\right]x^2$$

$$= \frac{1}{11}\left[1-\frac{D^2}{11}-\frac{6D}{11}+\frac{36D^2}{121}\right]x^2$$

$$= \frac{1}{11}\left[x^2-\frac{2}{11}-\frac{12x}{11}+\frac{72}{121}\right]$$

$$= \frac{1}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

Substituting in (1)

$$y_p = e^x \cdot \frac{1}{4}(\sin x+\cos x) + \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$= c_1\cos(\sqrt{2}x)+c_2\sin(\sqrt{2}x)+\frac{e^x}{4}(\sin x+\cos x)$$

$$+\frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right]$$

EXAMPLE-4: $(D^2-1)y = x^2\sin 3x$

Sol^n

Sol^n:- A.E is $m^2-1=0$

$$m=\pm 1$$

$\therefore$ C.F is $y_c = c_1 e^x + c_2 e^{-x}$

$$y_p = \frac{1}{D^2-1}(x^2 \sin 3x)$$

$\left[e^{i3x} = \cos 3x + i \sin 3x \right]$

$$= \text{I.P of } \frac{1}{D^2-1}\left(x^2 e^{3ix}\right)$$

$$\left[\frac{1}{f(D)} e^{ax}.V = e^{ax}.\frac{1}{f(D+a)} V. \right]$$

$$= \text{I.P. of } e^{i3x} \cdot \frac{1}{(D+3i)^2-1} \cdot x^2$$

$$= \text{I.P of } e^{i3x} \cdot \frac{1}{D^2+6iD-10} x^2$$

$$= \text{I.P. of } e^{i3x} \cdot \left(\frac{-1}{10}\right) \frac{1}{\left[1-\left(\frac{D^2+6iD}{10}\right)\right]} x^2$$

$$= \text{I.P of } \left(\frac{e^{i3x}}{-10}\right)\left[1-\left(\frac{D^2+6iD}{10}\right)\right]^{-1} x^2$$

$$\left[(1-t)^{-1} = 1+t+t^2+t^3+\cdots\cdots \right]$$

$$= \text{I.P of } \left(-\frac{e^{i3x}}{10}\right)\left[1+\left(\frac{D^2+6iD}{10}\right)+\left(\frac{D^2+6iD}{10}\right)^2+\cdots\right] x^2$$

$$= I\cdot P \text{ of } \left(\frac{-e^{i3x}}{10}\right)\left[1+\frac{D^2}{10}+\frac{3iD}{5}-\frac{36}{100}D^2\right]x^2$$

$$= I\cdot P \text{ of } \left(\frac{-e^{i3x}}{10}\right)\left[x^2+\frac{2}{10}+\frac{3i(2x)}{5}-\frac{72}{100}\right]$$

$$= I.p. \text{ of } \left(-\frac{e^{i3x}}{10}\right)\left[\left(x^2-\frac{52}{100}\right)+i\left(\frac{6x}{5}\right)\right]$$

$$= I\cdot P \text{ of } \left(\frac{-1}{10}\right)\left(\cos 3x+i\sin 3x\right)\left[\left(x^2-\frac{52}{100}\right)+i\left(\frac{6x}{5}\right)\right]$$

$$y_p = \left(\frac{-1}{10}\right)\left[\frac{6x}{5}\cos 3x+\left(x^2-\frac{13}{25}\right)\sin 3x\right]$$

The complete solution is

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{-x} - \frac{1}{10}\left[x^2\sin 3x + \frac{6x}{5}\cos 3x - \frac{13}{25}\sin 3x\right]$$

Example-5: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = xe^{-x}\cos x$

$$(D^2+2D+1)\,y = xe^{-x}\cos x$$

Soln:- A.E is $m^2+2m+1=0$

$$(m+1)^2=0 \qquad m=-1,-1$$

$\therefore$ C.F is $y_c = c_1 e^{-x} + c_2 x e^{-x} = (c_1+c_2x)e^{-x}$

$$y_p = \frac{1}{(D+1)^2}\, xe^{-x}\cos x$$

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\cdot\frac{1}{f(D+a)}V$$

$$y_p = \frac{1}{(D+1)^2} x e^{-x} \cos x$$

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \cdot \frac{1}{f(D+a)} V$$

$$= e^{-x} \frac{1}{(D-1+1)^2} x \cos x$$

$$\frac{1}{D} \rightarrow \text{Integration}$$

$$= e^{-x} \cdot \frac{1}{D^2} x \cos x$$

$$= e^{-x} \cdot \frac{1}{D} \int x \cos x \, dx$$

$$= e^{-x} \cdot \frac{1}{D} \left[x \sin x - \int (1) \sin x \right]$$

$$= e^{-x} \cdot \frac{1}{D} \left[x \sin x + \cos x \right]$$

$$= e^{-x} \cdot \int (x \sin x + \cos x) \, dx$$

$$= e^{-x} \left\{ x(-\cos x) - \int (1)(-\cos x) \, dx + \sin x \right\}$$

$$= e^{-x} \left\{ -x \cos x + \sin x + \sin x \right\}$$

$$y_p = e^{-x} \left\{ 2 \sin x - x \cos x \right\}$$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) e^{-x} + e^{-x} \left\{ 2 \sin x - x \cos x \right\}$$