EXACT DIFFERENTIAL EQUATIONS:

Definition : A differential equation which is obtained from its primitive differentiation only and without any operation of elimination or reduction is called an **exact differential equation**.

If $u = c$ where u is a function of x and y is primitive then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$ is an exact differential equation. Thus, an exact differential equation is obtained from its promitive by equating its total differential to zero.

For example, If $u = x^2 + y^2 = c$ then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 2xdx + 2ydy$

Equating $du = 0$, we get the equation $x\,dx + y\,dy = 0$ which is exact.

The Necessary and Sufficient Condition for equation $\boldsymbol{Mdx + Ndy = 0}$ to be exact is $\boldsymbol{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

i.e. If the equation $M\,dx + N\,dy = 0$ is exact, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and conversely if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is exact.

Rule for finding the solution:

Rule 1: Integrate M w.r.t x treating y constant and Integrate only those terms in N which are free from x w.r.t. y. Equate the sum to a constant. This is the solution.

In symbols, $\int M\,dx$ (treating y constant) + $\int (Terms$ in N free from $x)\,dy = c$.

Rule 2: Integrate N w.r.t y treating x constant and Integrate only those terms in M which are free from y w.r.t. x. Equate the sum to a constant. This is the solution.

In symbols, $\int N\,dy$ (treating x constant) + $\int (Terms$ in M free from $y)\,dx = c$.

EXAMPLES:

1. $x\,dx + y\,dy = \frac{a(x\,dy - y\,dx)}{x^2+y^2}$

Solution: The equation can be written as $\left\{x + \frac{ay}{x^2+y^2}\right\}dx + \left\{y - \frac{ax}{x^2+y^2}\right\}dy = 0$

$\therefore M = x + \frac{ay}{x^2+y^2}, \quad \therefore \frac{\partial M}{\partial y} = \frac{a}{x^2+y^2} - \frac{2ay^2}{(x^2+y^2)^2} = \frac{ax^2-ay^2}{(x^2+y^2)^2}$

$\therefore N = y - \frac{ax}{x^2+y^2}, \quad \therefore \frac{\partial N}{\partial x} = -\frac{a}{x^2+y^2} + \frac{2ax^2}{(x^2+y^2)^2} = \frac{ax^2-ay^2}{(x^2+y^2)^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact

Now, $\int M\,dx = \int x\,dx + ay\int \frac{dx}{x^2+y^2} = \frac{x^2}{2} + ay\cdot\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) = \frac{x^2}{2} + a\cdot\tan^{-1}\left(\frac{x}{y}\right)$

And $\int(\text{terms in } N \text{ free from } x)\,dy = \int y\,dy = \frac{y^2}{2}$

$\therefore$ This solution is $\frac{x^2}{2} + \frac{y^2}{2} + a\tan^{-1}\left(\frac{x}{y}\right) = c$ i.e. $x^2 + y^2 + 2a\tan^{-1}\left(\frac{x}{y}\right) = c$

2. $2\left(1 + x^2\sqrt{y}\right)ydx + \left(x^2\sqrt{y} + 2\right)x\,dy = 0$

Solution: Here, $M = 2y + 2x^2y^{3/2}; N = x^3\sqrt{y} + 2x$

$\therefore \frac{\partial M}{\partial y} = 2 + 3x^2y^{1/2}; \frac{\partial N}{\partial x} = 3x^2\sqrt{y} + 2 \qquad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The equation is exact

Now, $\int M\,dx = \int\left(2y + 2x^2y^{3/2}\right)dx = 2xy + \frac{2}{3}x^3y^{3/2}$

And $\int(\text{terms in } N \text{ free from } x)\,dy = \int 0\cdot dy = 0$

$\therefore$ The solution is $2xy + \frac{2}{3}x^3y^{3/2} = c$

3. $\frac{y}{x^2}cos\left(\frac{y}{x}\right)dx - \frac{1}{x}cos\left(\frac{y}{x}\right)dy + 2xdx = 0$

Solution: We have $\left[2x + \frac{y}{x^2}\cos\left(\frac{y}{x}\right)\right]dx + \left[-\frac{1}{x}\cos\frac{y}{x}\right]dy = 0$

$\therefore M = 2x + \frac{y}{x^2}\cos\frac{y}{x}$ and $N = -\frac{1}{x}\cos\frac{y}{x}$

$\therefore \frac{\partial M}{\partial y} = \frac{1}{x^2}\cos\left(\frac{y}{x}\right) - \frac{y}{x^2}\sin\left(\frac{y}{x}\right)\cdot\frac{1}{x}$; $\frac{\partial N}{\partial x} = \frac{1}{x^2}\cos\left(\frac{y}{x}\right) - \frac{1}{x}\sin\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact

$\therefore \int M\,dx = \int\left(2x + \frac{y}{x^2}\cos\frac{y}{x}\right)dx = \int 2x\,dx + \int\cos\left(\frac{y}{x}\right)\cdot\frac{y}{x^2}dx = x^2 + I_2$

For I_2, put $\frac{y}{x} = t, -\frac{y}{x^2}dx = dt$

$\therefore \int M\,dx = x^2 - \int\cos t\,dt = x^2 - \sin t = x^2 - \sin\left(\frac{y}{x}\right)$

$\int$(terms in N free from x) $dy = 0$

$\therefore$ The solution is $x^2 - \sin\frac{y}{x} = c$

4. $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0, given\ y(0) = 4$

Solution: Here $M = 1 + e^{x/y}, N = e^{x/y}\left(1 - \frac{x}{y}\right)$

$\therefore \frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$; $\frac{\partial N}{\partial x} = e^{x/y}\cdot\frac{1}{y}\left(1 - \frac{x}{y}\right) - e^{x/y}\left(\frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The equation is exact

$\therefore \int M\,dx = \int\left(1 + e^{x/y}\right)dx = x + ye^{x/y}$

$\int$(terms in N free from x) $dy = \int 0\,dy = 0$

$\therefore$ The solution is $x + ye^{x/y} = c$

By data when $x = 0, y = 4$ $\therefore 4 = c$

The particular solution is $x + ye^{x/y} = 4$